4 Symmetry in music

A GEOMETRIC OBJECT or shape is symmetric if it can be divided into two or more identical parts and is invariant under transformation. In other words, it can be transformed by moving individual pieces whilst keeping the same overall shape. There are several different types of transformation; the main types are reflection, rotation, translation, and scale. Much Western music has its basis in repetition and contrast so it is not surprising, given that symmetry is a form of repetition, that it is frequently found in music. As Arnold Schoenberg (1874-1951) wrote in *Fundamentals of Musical Composition* (1967) 'A melody, classic or contemporary, tends towards regularity, simple repetitions and even symmetry.'[72] The repetition of ideas helps bind music together. One idea is not enough, but if there are too many ideas following each other with no unifying factor then the music may sound shapeless and without a sense of direction. Symmetry is a way of creating unifying factors. This chapter looks at the different ways in which musicians have used symmetry as a formal element from the Medieval composer Guillaime de Machaut (c 1300-1377) in 'Ma fin est mon commencement et mon commencement ma fin' to Nirvana in 'Smells like teen spirit'.

72 Arnold Schoenberg. *Fundamentals of Musical Composition* (London: Faber & Faber, 1982): 103.

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Reflection

When an object has reflectional symmetry it can be divided into two pieces which are images of each other. The dividing line is known as the axis of symmetry. This could be the on the horizontal x axis or the vertical y axis. The letters B, C, D, E and K all reflect along the x axis. This is known as a vertical reflection. Or the reflection could be along the vertical y axis. This is referred to as horizontal reflection and it is sometimes referred to as temporal reflection symmetry. The letters A, M, T, U, V, W, M and Y all reflect along the vertical y axis. Melodies can therefore be reflected in two ways: along the x axis and along the y axis.

Inversion

When music is reflected along the x axis, what went up goes down and vice versa. This is known in music as inversion. It can be done in two different ways or a combination of both.

1. The intervals are inverted so that only the numbering of the interval is preserved: E down to C (a falling 3rd) becomes E up to G (a rising 3rd).

2. The exact intervals are inverted so that a major 3rd down becomes a major 3rd up. E down to C (a falling major 3rd) becomes E up to G# (a rising major 3rd).

Reflection is clear to see in the outer parts of the left hand (LH) and right hand (RH) of Bartók's piano piece *Mikrokosmos No.* 141. Béla Bartók (1881-1945) wrote *Mikrokosmos* in 1939. It is a series of 153 piano pieces ranging in difficulty from easy to advanced with each one featuring a different scale or mode, rhythmic or melodic feature. In No. 141 'Subject and Reflection' the LH part is an exact reflection of the RH part (see below). In this example the theme and its inversion are played simultaneously.

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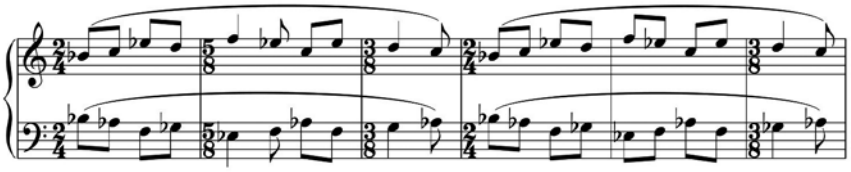
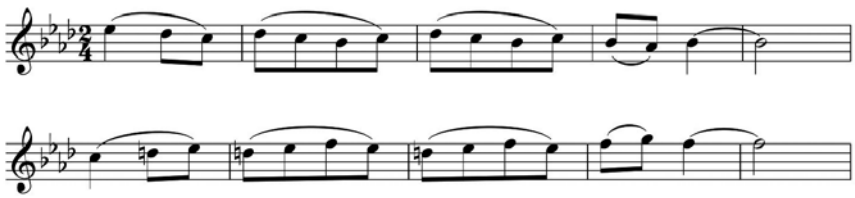


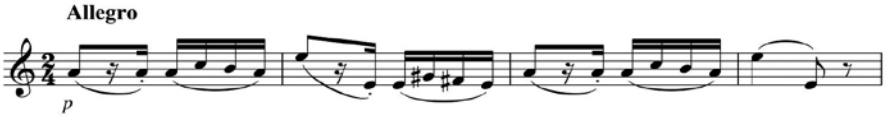
In the first bar Bb up to C in the top part (a major 2nd) becomes Bb down to Ab (a major 2nd), then C up to Eb in the upper part (a minor 3rd) becomes Ab down to F (a minor 3rd). Notice how the number of this piece, No. 141 is also symmetric. Although virtually all of No. 141 demonstrates this simultaneous reflection, later on in the piece there are a few bars where one of the notes in the inner parts is deleted, a compromise to fit in with Bartók's harmonic scheme.

In the third movement *of Symphony No. 1* (1876) by Johannes Brahms (1833-1897), a different approach is taken where the second half (bars 6 - 10) of the opening clarinet melody is a mirror image of the first half (bars 1 – 5) (see below).

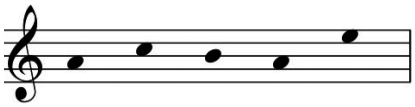


In the first bar, Eb down to Db (a major 2nd) becomes C to D (a major 2nd) in the sixth bar, then Db down to C (a minor 2nd) becomes D natural to Eb (a minor 2nd) and so on. Notice how D flat has to become D natural in order to maintain the same intervals.

One of the most famous examples of the use of inversion can be found in the *Rhapsody on a Theme of Paganini* for piano and orchestra by Sergei Rachmaninov (1873-1943). The work is a set of 24 variations based on Niccolò Paganini's 24th *Caprice* for solo violin. Here is the main theme which is heard in the strings towards the beginning of the piece (see below):

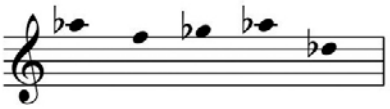


Original



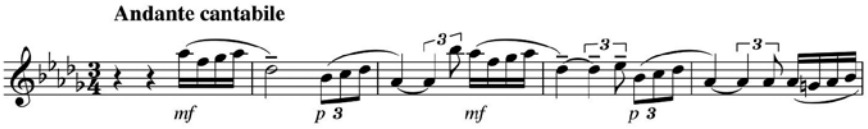
A to C - up a minor 3rd
C to B - down a minor 2nd
B to A – down a major 2nd
A to E – up a perfect fifth

Variation XVIII



Ab to F -down a minor 3rd
F to Gb - up a minor 2nd
Gb to Ab – up a major 2nd
Ab – Db – down a perfect fifth

Variation XVIII is based on an inversion of the main semiquaver – quaver figure found in the first bar of the main theme with all the intervals preserved (see above). The tempo is slowed down and there is a change of key. The result is a complete change of character where the original spritely theme is transformed into a languorous romantic melody.

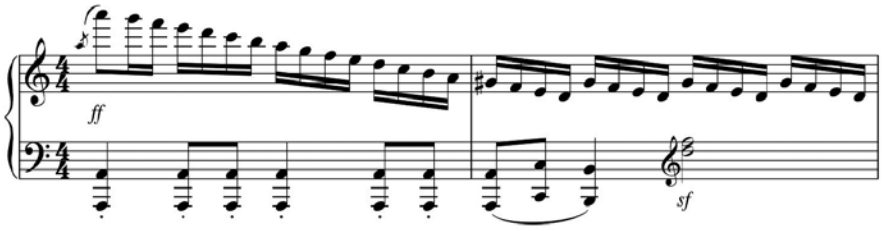


The transformation is very effective, but it is difficult to discern by the ear alone.

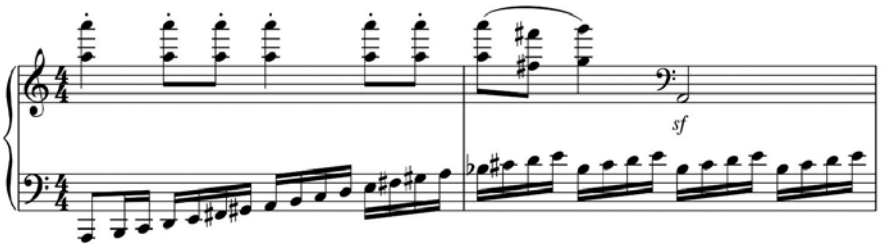
In contrast, the inversion used by Franz Schubert (1797-1828) in his Fantasia Op. 15[73] is easily discernible: the descending scale becomes an

73 Schubert's Fantasia Op. 15 is often known as the *Wanderer Fantasia* because of its use of the melody taken from his lied 'Der Wanderer'.

ascending scale and the chords move from the bass part to the treble part (see below).



Franz Schubert· *(Wanderer) Fantasia Op. 15* (bars 83-84)



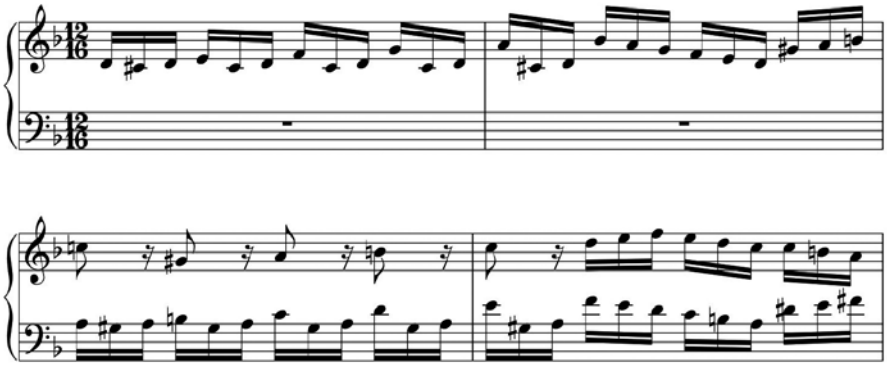
Franz Schubert· *(Wanderer) Fantasia Op. 15* (bars 90-91)

There is often an element of compromise in the use of inversion – or indeed any of the symmetrical devices. This is because of the musical constraints imposed by tonal music and functional harmony. In order to avoid unwanted dissonance or tonal ambiguity, precise intervals are not always used: a major third may become a minor third, for example.

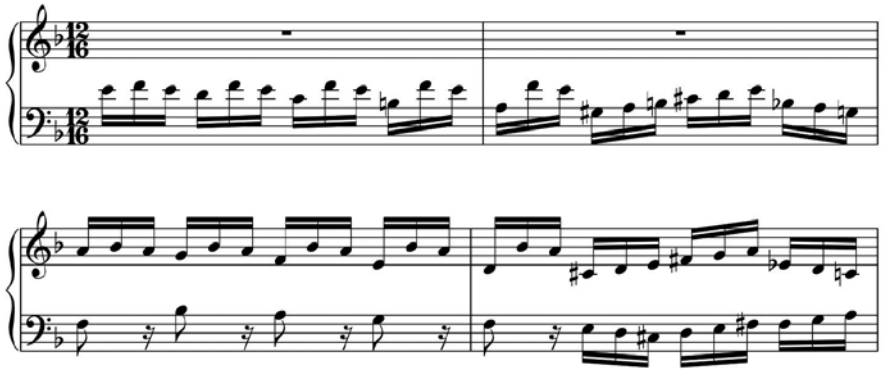
The question remains as to whether composers were consciously using points of symmetry in their works. It is not always possible to know their intentions. In the above examples it is fair to assume that Bartók was consciously using inversions, given the title 'Subject and Reflection'.

J S Bach (1685-1750) makes extended use of inversion in the Gigue of his *English Suite No. 6 in D minor* and it would appear that this was intentional. When the second section is compared with the first, we can see that the latter

is a mirror version of the former (see below). This is all the more impressive because of its chromatic nature.



J S Bach· English Suite No. 6 in D minor - Gigue (opening)



J S Bach· English Suite No. 6 in D minor - Gigue (opening of second section)

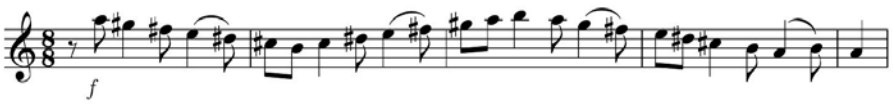
Furthermore this mirror image is maintained for most of the second part of the Gigue. This cannot have been an accident: it must have been intentional. Further evidence of this is given by Bach's treatment of the form. During the Baroque period, the gigue was one of the most popular instrumental dances and a standard movement of the suite. Most gigues were in binary form. One of the features of two-section binary form is that the first section, the A

section, opens in the tonic key (in this case D minor) and closes in another key, usually the dominant or the relative major (in this case A major). The B section opens in the new key and modulates back to the tonic. In order to comply with the conventions of binary form, Bach inserts eight extra bars which do not follow the mirror pattern shortly before the end of the piece.

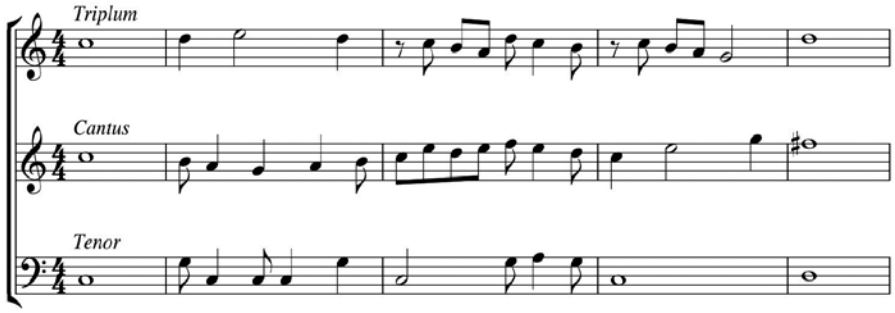
Retrograde

In horizontal reflection, the last note becomes the first and the notes appear in reverse order, back to the first. Sometimes the rhythm is retained, at other times the rhythm is abandoned. When music is reflected along the vertical y axis the notes form a kind of palindrome. This is known in music as retrograde.

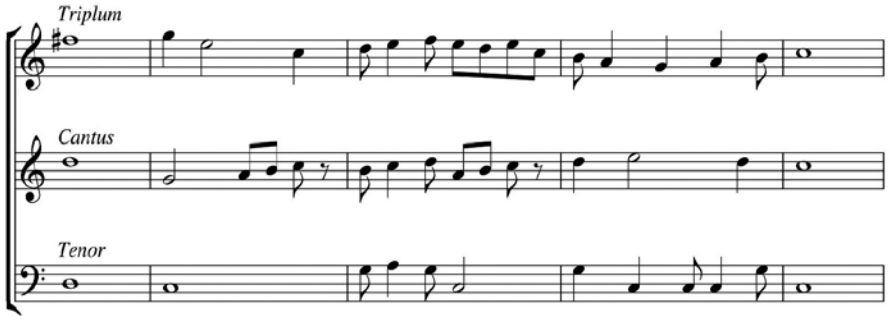
Béla Bartók uses retrograde motion in the fourth movement of his 1936 work *Music for strings, percussion and celesta*. Here the piece is in a Bulgarian folk style with a rhythm pattern of 2 + 3 + 3 beats. Quavers and crotchets are used for both the opening melodic line (A G# F# E D# C# B, notes 1 – 7) and its reflection (B C# D# E F# G# A notes 7 – 13) but the notes fall on different strong or weak beats (see below).



The use of retrograde was very popular in Medieval and early Renaissance music when it was sometimes known as *cancrizans*, the Latin form of crab (although it should be noted that crabs walk sideways most of the time). One of the most famous examples is by the Medieval French composer Guillaume de Machaut. In his three-voice setting of 'Ma fin est mon commencement et mon commencement ma fin' (My end is my beginning and my beginning is my end) the *triplum* part was created by reading the *cantus* line in reverse. Here are the opening bars….



… and the closing bars where the last four bars of the cantus are an exact reversal of the first four bars of the triplum in terms of both pitch and rhythm.

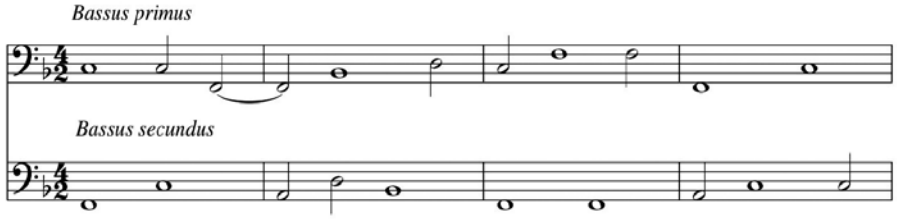


How does this this sound? As the musicologist William Drabkin observes of the passage, it has been 'admired more for the finesse with which the technique is used than for the chanson's other artistic merits'.[74]

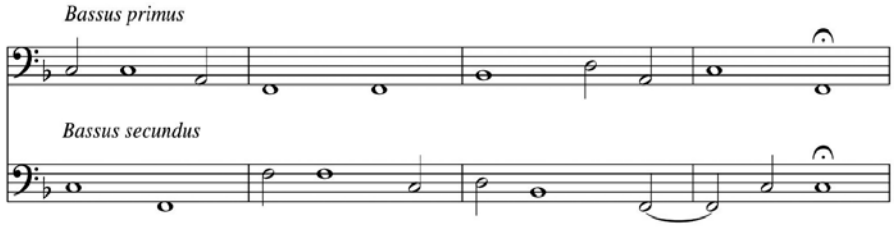
In 1597 Thomas Morley (1557-1602) published his famous musical treatise *A Plaine and Easie Introduction to Practicall Musicke* which includes instructions on how to compose a palindromic piece for voices. He writes '… you may make eight partes in foure (or fewer or more as you list) which may be sung backward and forward, that is, one beginning at the beginning of every part, and another at the ending, and so sing it quite through' He goes on to set out some (rather confusing) rules. He warns against using dotted notes because although 'in singing the part forward, it will go well' when coming backwards it 'will make a disturbance in the music because the singer

74 William Drabkin. 'Cancrizans' in *Grove Music*.

will be in a doubt to which note the dot belongeth'. He includes an example of an eight part canon (for more about canons see pages 141-52) and then writes 'If you desire more examples of this kind, you may find one of master Byrd, being the last song of those Latin Motets which under his and master Tallis his name were published'.[75] The piece Morley refers to is the eight-voice motet *Diliges Dominum* (1575) by the composer William Byrd (1540-1623) who had once taught Morley.[76] It is a musical palindrome for double choir where one four-part choir (soprano, alto, tenor, bass) sings the same music as the other choir, exactly in reverse; at the halfway point the two voices of each pair exchange parts and present them backwards. Here are the four opening bars of the 'bassus primus' part – the bass in choir 1.



… and here are the four closing bars of the bassus secundus part - the bass in choir 2. The bassus primus part of the opening is now sung in reverse order by the bassus secundus, and the opening notes of the bassus secundus are sung in reverse order by the bassus primus.

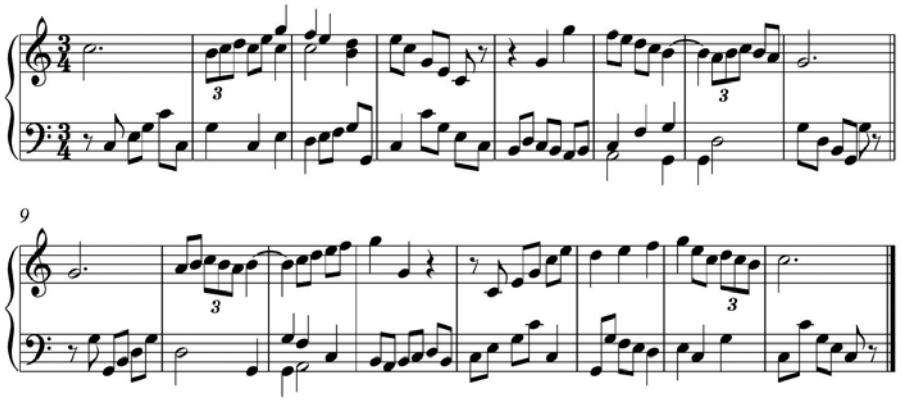


75 Thomas Morley. *A Plaine and Easie Introduction to Practicall Musicke* (London, 1597): 175-6.

76 *Diliges Dominum* is part of the Cantiones Sacrae (Sacred Songs), a collection of works by William Byrd and his teacher Thomas Tallis dedicated to Queen Elizabeth I.

It has been argued that the retrograde used here was in response to the text and that the exhortation to 'love thy neighbour as thyself' occasioned the reprise to bring the music full circle.[77]

Nearly 200 years later C P E Bach (1714-1788) composed his Minuet in C (1770) for keyboard. This is made up of two eight-bar sections where the second section is reversal of the first.



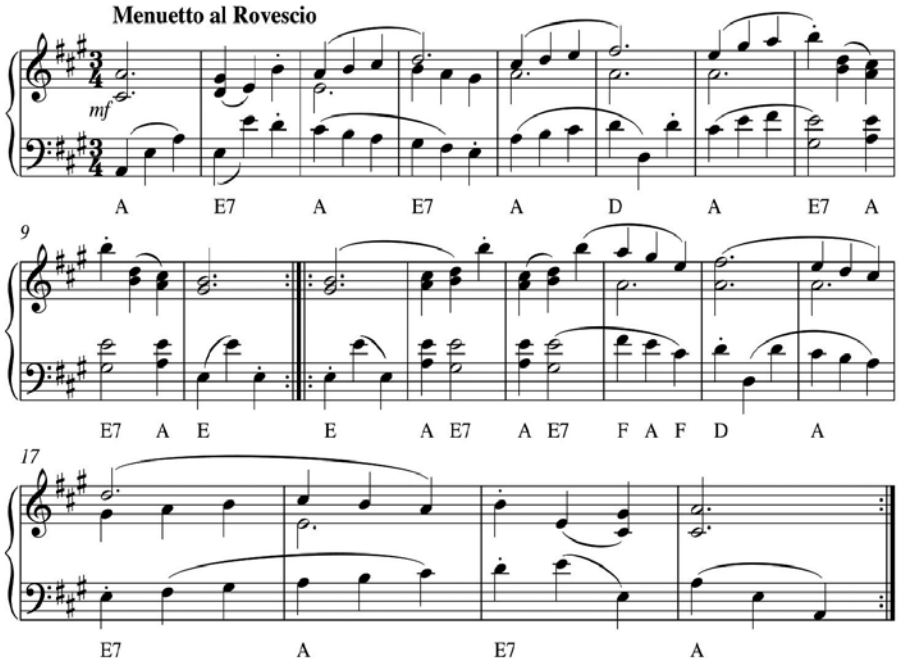
The piece is short and simple and is largely built on two chords, C and G, (the tonic and dominant/dominant 7th in the key of C). This means that when played in reverse the chords and harmony stay the same, a means of avoiding unwanted dissonances or tonal ambiguity.

Two years later Joseph Haydn (1732-1809) included a 'Menuet al Reverso' (minuet in reverse) in his Symphony No. 47 in G hence the symphony's nickname 'The Palindrome'. The second half of the minuet is identical to the first, but it is played in reverse - backwards. The trio has a similar palindromic construction. It is no accident that both C P E Bach and Haydn both chose to use a minuet to employ this palindromic device. Minuets during this period were usually light-hearted dance movements, less sophisticated in terms of both harmonic rhythm and texture in comparison with the more serious musical business taking part in the first and second movements of the symphonies and sonatas that they often formed part of.

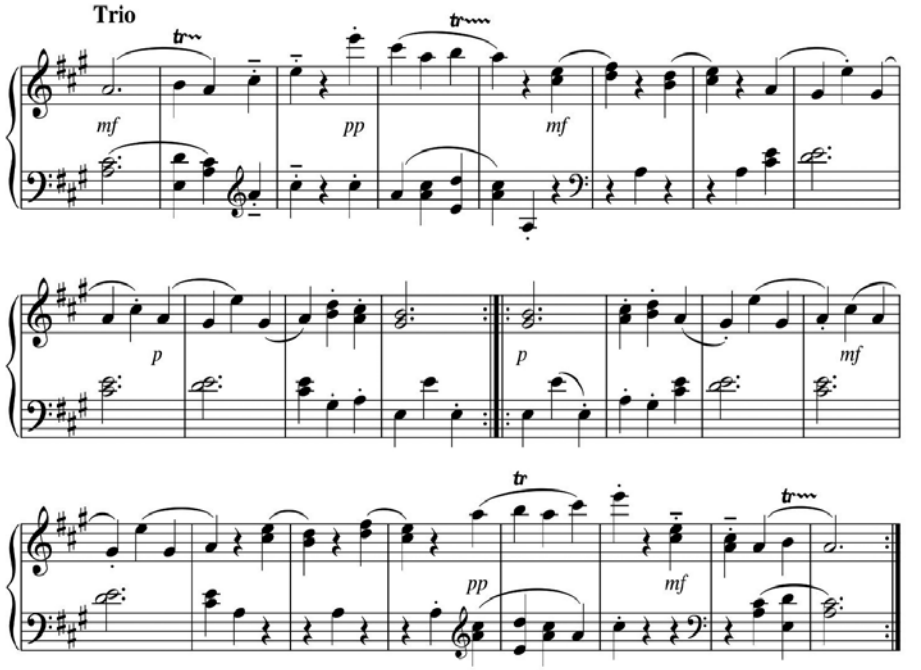
Haydn arranged the Minuet and Trio from his Symphony No. 47 for his Piano Sonata No. 26 in A. Here it is easy to see the vertical reflection,

77 Brian Newbould. 'A Schubert palindrome'. *19th-Century Music* Vol. 15, No. 3, 1999): 207-214.

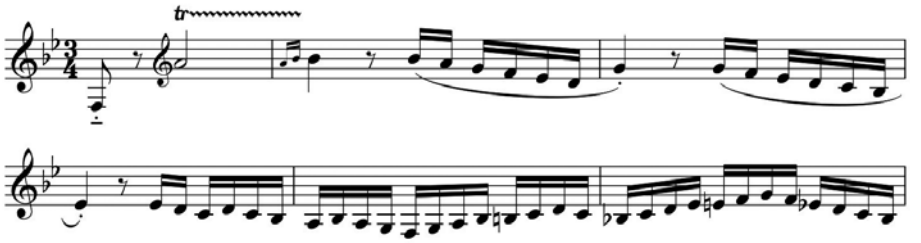
between bars 10 and 11 in what he now names the 'Menuetto al Rovescio'. In the same way as that of C P E Bach, Haydn's harmonic scheme for both the Menuet and Trio is firmly centred on the chords of A and E/E^7 major the tonic and dominant/dominant 7th. This means that when played in reverse, the harmony stays the same (see below).



The Trio uses a similar combination of tonic and dominant seventh chords (see below). Notice how both the left and right hand parts play their parts backwards. The point of symmetry is between bars 12 and 13 (see below).

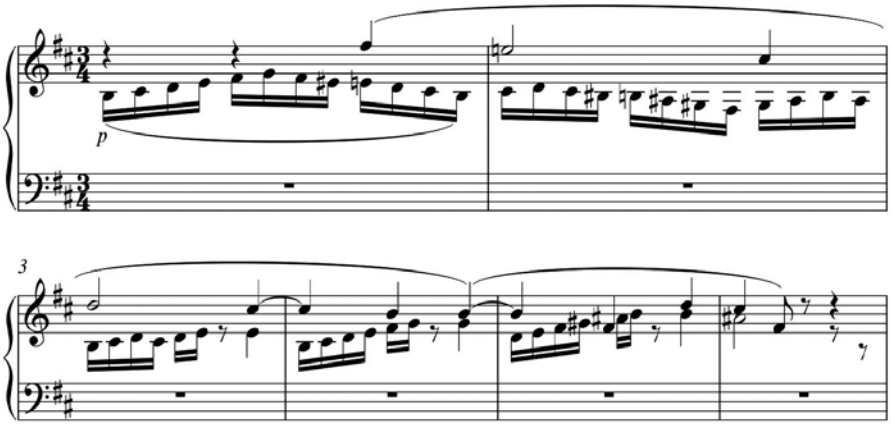


Another famous example of the use of retrograde can be found in the fugue of the fourth movement of the *Hammerklavier Sonata, Op. 106* by Ludwig van Beethoven (1770-1827) which was composed in 1818. Here is the first announcement of the theme:



Ludwig van Beethoven· *Hammerklavier Sonata Op. 106* Movt. IV (bars 16-21)

In the earlier examples by C P E Bach and Haydn, the retrograde version of the theme follows the original immediately whereas the retrograde version of Beethoven's theme appears 137 bars later. The backward rhythm is exact, but it is now in B minor rather than Bb major. Any problems with the harmonic scheme are avoided by the fact that the first appearance, unlike the second, is not harmonised (although there is inevitably some implied harmony in the first).



Ludwig van Beethoven. *Hammerklavier Sonata Op. 106* Movt. IV (bars 153-158)

In his article 'A Schubert Palindrome', the musicologist Brian Newbould writes with surprise of his discovery of what he considers to be the most ambitious example of a nineteenth-century musical palindrome. It is a passage in Schubert's little-known opera-melodrama *Die Zauberharfe* (The Magic Harp, 1820). The passage is long and surprisingly complex given that it comes from the pen of a composer who compositional process is often considered as 'intuitive' with 'finitely shaped gems' arriving in 'an instant'. As Newbould writes 'Schubert is one of the least likely perpetrators of the cerebral deed'.[78] The passage can be found near the opening of No. 3 with the retrograde version following the original 309 bars later.[79] Newbould describes it as a 'technical tour de force' where the 'harmonic thinking is far more venturesome

78 Newbould, 'A Schubert palindrome', 214.

79 *Die Zauberharfe* was not a critical success and after only eight performances it disappeared from the repertoire. However, after Schubert's death the overture was published as the overture to *Rosamunde*.

than that in the Haydn minuet'. Furthermore it is written for large Romantic orchestral forces and, at 19 bars long, is significantly longer than the C P E Bach and Haydn examples. At the same time Newbould notes that in order to comply with tonal constraints, Schubert used a certain amount of licence in the ordering of pitches and rhythms within a bar or half-bar. There are two main ways in which he deviates: firstly, at times the reversal of pitches is not accompanied by the reversal of rhythm; and secondly, sometimes the two halves of a bar may change places, but the original order of notes (rather than the reverse order) is retained within each half bar. Nevertheless, as Newbould writes, although the palindrome is not exact 'only the most keen-eared and thoughtful listener would detect the deviations from exactitude.'[80] Bars 7 – 25 are found in retrograde in bars 334 – 353. Here are the string parts for bars 21- 24 followed by the retrograde version found in bars 335 – 338 (see below).

80 Newbould, 'A Schubert palindrome', 211.

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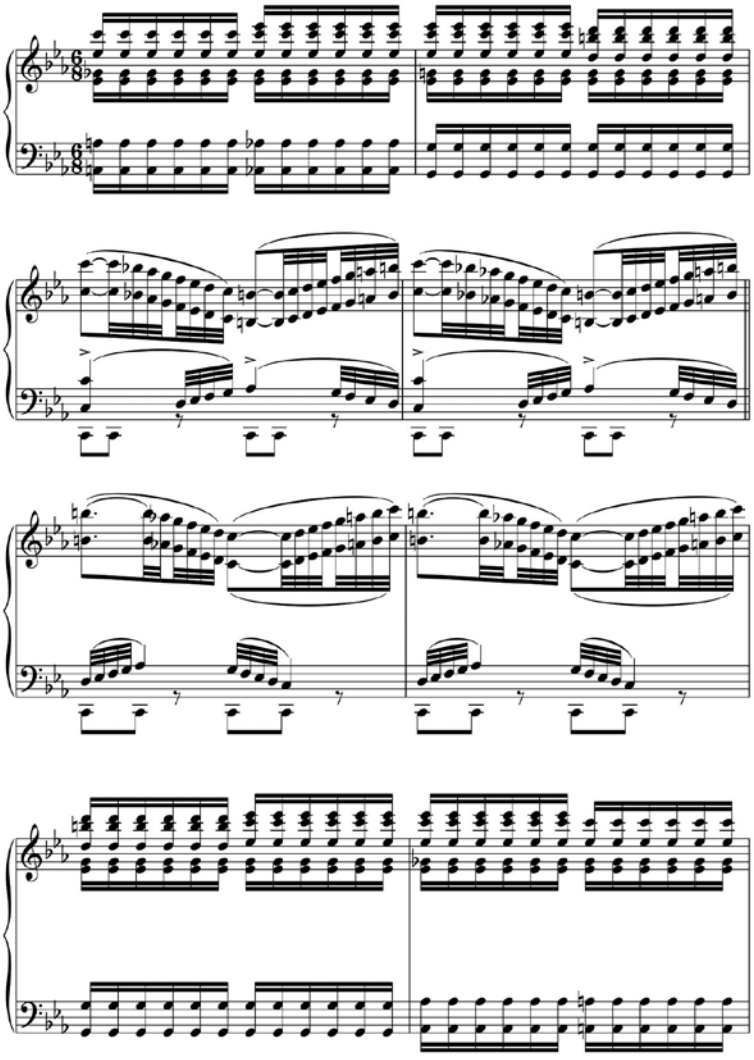

Franz Schubert· *Die Zauberharfe, No. 3* (strings, bars 21-24 followed by strings bars 335-338)

As we have seen, diatonic music and functional harmony can impose certain musical constraints when using retrograde devices, so, for example, both Haydn and C P E Bach restricted the harmony of their retrograde pieces to little more than two chords. The twentieth century however heralded new approaches to harmonic schemes and consequently more freedom for

composers when using palindromes. This was particularly evident in the music of the Second Viennese School where tonality was largely abandoned to make way for twelve-tone techniques as we will see in Chapter 8. For composers such as Arnold Schoenberg (1874-1951), Alban Berg (1885-1935) and Anton Webern (1883-1945) the advent of serialism considerably eased the strict harmonic demands of palindromic music.

Rotation

When an object has rotational symmetry it can be turned about a fixed point whilst keeping the same overall shape. One way this can be done is by reflecting the figure across both axes. Reflection across the x axis results in inversion and reflection across the y axis results in retrograde, so when the two types of reflection are put together the result is retrograde inversion. Some composers combine both inversion and retrograde – the music is both backwards and upside down. This is known as retrograde inversion. Its geometric equivalent is 180^0 rotation.

Paul Hindemith's 1942 piano work *Ludus Tonalis* ('Tonal Play') is subtitled 'Studies in Counterpoint, Tonal Organization and Piano Playing'. The Tonal Play of the title could well refer to the way that the composer plays with symmetries: different types of reflection and a symmetrical layout. It is one of few pieces to use rotation. The work opens with a three-part Praeludium and ends with a Postludium. The Postludium is a rotated version of the opening movement: visually it is an exact retrograde inversion of the Praeludium (with an added final chord). That is, the Postludium is the Praeludium turned upside down and played backwards.

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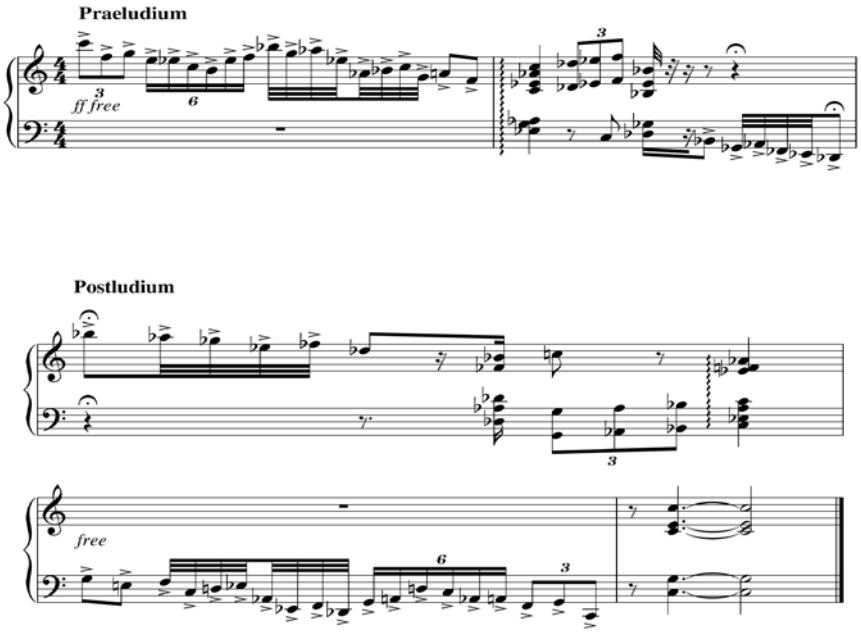
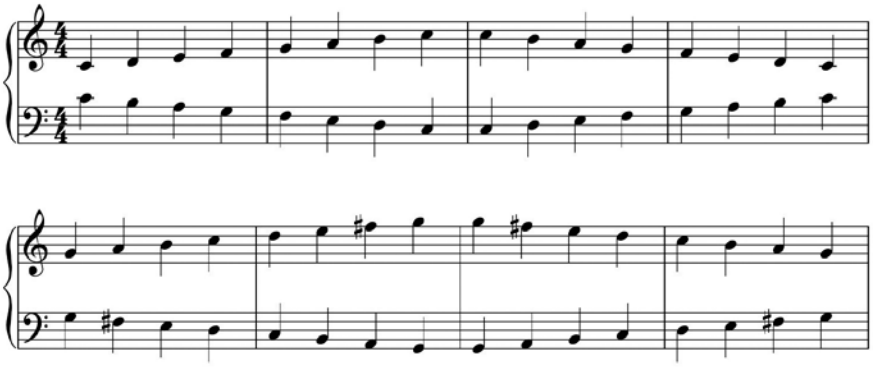


If we focus on the first bar of the Praeludium we can see that the equivalent bar of the Postludium is played backwards in terms of note values, so that:

quaver triplet, semi-quaver sextuplets, eight demi-semi-quavers then two quavers *becomes* -
two quavers, eight demi-semi-quavers, semi-quaver sextuplet then quaver triplets.

The melodic inversion here however, is visual rather than musical. Rather than strictly inverting the intervals used, the notes are selected according to how they look upside down on the page and in the bass clef (allowing adjustment for the accidentals which must come before the note). In order for the movement to work, there were several things to take into consideration. We know this from the numerous sketches that Hindemith (1895-1963) made for this movement. It needed to make musical sense rather than simply be a technical exercise so, for example, dynamics and marks of expression had to remain appropriate in the visual retrograde inversion. Also, as Hindemith discovered, there were only five scales – C, C# and Cb major,

and the Phrygian mode on C and C# - that would resemble a mirror image when the pitches were inverted and placed in the bass clef. As can be seen below, the note F# spoils the mirror image pattern in the scale of G.



Mirror images of the scales of C major and F major

Retrograde inversion is much more uncommon than the other transformations used in music composed before the twentieth century, but it is an important element of the compositional technique known as serialism or twelve-tone technique (see Chapter 8) which first appeared in the 1920s.

Translation

When an object has translational symmetry it can be shifted a fixed distance in a fixed direction whilst keeping the same overall shape. This can happen in two ways.

1. It can be shifted from left to right – horizontal translation * * *
OR
2. It can be shifted upwards or downwards – vertical translation.
*
*
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Horizontal translation is essentially repetition and there are many examples in music where the same bar is repeated one or more times. A well-known example can be found in the repeated snare drum pattern used in the orchestral piece *Bolero* by Maurice Ravel (1875-1937) (see p.84). Electronic Dance Music (EDM) makes much use of horizontal translation and is largely based on tape loops of short repeated rhythms. A feature of pop music in general is its use of repeated chord progressions, drum patterns, and riffs. A riff is a short, repeated melodic or harmonic pattern usually of two or four bars' duration, although it may be heard at different pitches to fit in with the harmony, it is often simply repeated over and over as in the famous opening riff of the Nirvana song 'Smells like teen spirit' (1991). 'Smells like teen spirit' uses basically the same riff and chord progression (F- Bb – Ab – Db) throughout; the structure is primarily defined by differences in instrumentation and dynamics with the pattern of quiet verses building to loud, powerful choruses.

When an extended sequence of notes is taken and each note is shifted by the same interval, up or down in vertical translation, then this can result in a key change or modulation, something that happens all the time in music. A common device in pop music is to raise the key a tone or a semitone higher for the last verse or chorus of a song. This gives a sense of climax and allows the singer to hit higher notes. Michael Jackson's 'Man in the Mirror' (1988), for example, features a semitone key change at the repeat of the final chorus. Ella Fitzgerald takes this to new extremes in the famous 1960 recording of the Kurt Weill song 'Mack the Knife', where she forgets the words. The songs opens in the key of G and then each verse is taken up a semitone (G – Ab – A – Bb – B – C) and finally ending in the key of Db.

On a smaller scale, a short melody or harmonic progression can be repeated (more or less exactly) at a different pitch either higher or lower. This is termed a 'sequence' and is used in countless pieces of music. One of the most famous examples of the use of sequence is in the first movement of Beethoven's Symphony No. 5.

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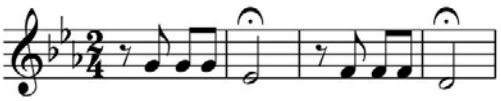
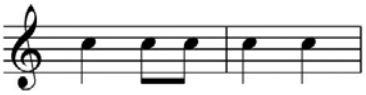


This repeated four-note cell or motif unifies the movement and propels it along in its different guises of repetition, inversion, transposition and variation.

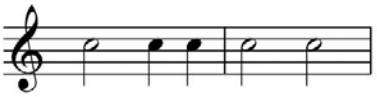
For more on the use of translation see the following chapter on Frieze Patterns.

Scale

An object has scale symmetry when it can be expanded or contracted whilst keeping the same overall shape.



The one-bar rhythm above can be transformed through scale symmetry in two ways: augmentation and diminution. The note values can be expanded – made longer - through augmentation. In this example the note values have been doubled.



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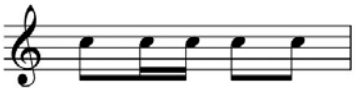
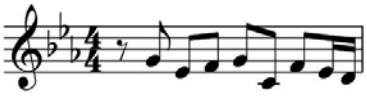
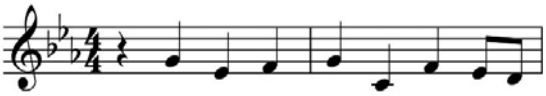
The note values can be contracted – made shorter - through diminution. In this example the note values have been halved.



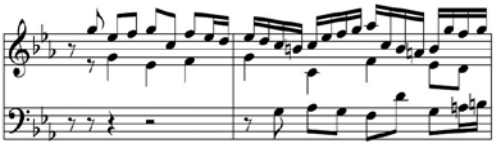
Augmentation and diminution are musical devices which are often found, though not exclusively, in fugues and canons (ssee Chapter 9). Here is an example of augmentation taken from J S Bach's Fugue No. 2 in C minor from Book II of the *Well-Tempered Clavier*. The fugue opens with the subject made up of quavers and semi-quavers:



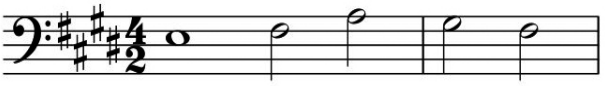
In bar 14 the note values of the subject are doubled, the quavers become crotchets and the semi-quavers become quavers:



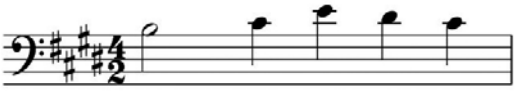
This is how it appears in the music as an inner-part.



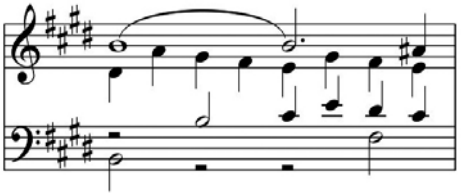
Here is an example of diminution. This is again taken from Book II of the *Well-Tempered Clavier*. Fugue No. 9 in E major opens with the subject in the bass. The first note values are a semibreve and four minims (see below).



In bar 28 it is heard in diminution – all the note values are halved.



Here it is *in situ* in an inner part.



Further examples of augmentation and diminution can be found in the chapters on 'Canons and fugues' and 'Serialism' where they are common features.

The next chapter looks at another aspect of symmetry; frieze patterns, plane figures with a symmetric pattern bounded by two infinite parallel lines.

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