6 Diatonic scales and tuning

THIS CHAPTER DESCRIBES various scales, intervals and systems of tuning. Most of the scales are commonly used, some of them less so. It opens by introducing the music theory involved; the diatonic major scale, intervals and the concept of consonance and dissonance. For centuries there has been much debate about different scales and tunings with many eminent mathematicians, scientists and musicians across Europe devising and analysing scales. What follows focuses on the three main systems that evolved; the Pythagorean scale, just intonation and equal temperament. It covers the mathematical and scientific principles involved such as ratios, square roots and irrational numbers, the harmonic series and the circle of 5ths.

The diatonic major scale, intervals, consonance and dissonance

A scale is a pattern of notes arranged in order of pitch from low to high (or vice versa) with specified distances or intervals between them. An interval is the distance between two notes and, in Western music, these are usually measured in numbers of tones or semitones. In this way from C up to D or down to B is a 2nd, another step from C up to E or down to A makes a 3rd, and so on.

Major and minor scales are known as diatonic scales and are built on a pattern of seven notes within an octave span. All major scales are built on the following pattern

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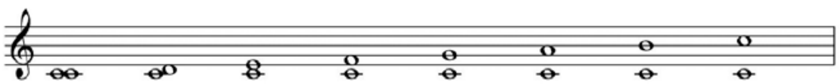
Tone Tone Semitone Tone Tone Tone Semitone

The semitones come between the 3rd and 4th degrees of the scale and the 7th and octave. Here is the scale of C major. It uses only the white notes on the keyboard.

V	V	V	V	V	V	V
tone	tone	semitone	tone	tone	tone	semitone

The major scale can be reproduced at any pitch i.e. it can start on any note. The following shows the names of all the intervals found between the tonic (key-note) of a major key and the other degrees of the scale. This serves as a useful standard by which other intervals can be worked out.



unison	major 2nd	major 3rd	perfect 4th	perfect 5th	major 6th	major 7th	octave

The character of different intervals is often referred to in terms of consonance and dissonance. Consonant intervals feel relatively stable and do not need to resolve to another interval. Major and minor 3rds and 6ths, perfect 5ths and octaves are all consonant intervals. Two voices producing the same pitch are said to be in unison. Dissonant intervals feel somewhat unstable, as though one of the notes needs to move up or down to resolve into a consonance. The major 7th, for example, feels as though it needs to resolve upwards, and the minor 7th feels as though it needs to resolve downwards. Major and minor

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2nds, perfect 4ths, major and minor 7ths and all augmented and diminished intervals are dissonant. Consonance is often loosely defined as being pleasing to the ear, dissonance being the antithesis, unstable and needing resolution. These definitions, however, should be treated with caution: the pleasing/displeasing notion depends on aesthetic preferences and implies a psychoacoustic judgment, whereas the notion of resolution of tension depends upon a familiarity with Western tonal harmony.

The Pythagorean scale

One of the earliest recorded musical scales was the Pythagorean scale. Pythagoras, the Greek mathematician and philosopher lived in the second half of the 6th century BCE. Although the system of construction of this scale existed long before his time, the term Pythagorean scale came about because of his theoretical justification in mathematical terms. Apparently, in passing a blacksmith's forge, Pythagoras heard different musical intervals in the striking of hammers against the anvils. Hammers of different weights struck simultaneously produced different consonant and dissonant intervals, so, for example, a hammer weighing half as much as another, a ratio of 2:1, produces a note an octave higher. Pythagoras went on to deduce from this that there was a relationship between consonant sounds and simple ratios and investigated further by carrying out a series of experiments with other bodies; water-filled glasses, strings, bells and pipes. The results pointed to unchanging relationships between the dimensions of the instruments used and the notes they produced. The ability to express these relationships numerically made it possible to analyse scales.

Using a monochord, a single-string instrument said to have been his invention, Pythagoras discovered that when a string is stopped half way the shorter string vibrates with twice the frequency and sounds an octave higher. Several intervals can be identified in the ways that they correspond to simple ratios of sound wavelengths or frequencies. So, for example, an octave is identified by the ratio 2:1 because the frequency of the upper note is twice that of the lower note, the interval of a 5th can be produced by the ratio 3:2 and a 4th by the ratio 4:3 From this the Pythagorean scale can

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be constructed by taking a note and producing others related to it through simple whole number ratios.[94]

In order to find the ratios of further intervals, Pythagoras multiplied two ratios together. As we have seen, an octave uses the ratio 2:1 and a 5th uses the ratio 3:2. Going up an octave and then down a 5th produces a 4th. We know that the 4th uses the ratio 4:3 and it follows that this can be calculated by 1/2 x 3/2 = 3/4. Using this principle, other degrees of the scale can be determined (see Table 1). So for example, if W = 9/8, then 1/1 x W = 9/8, 9/8 x W = 81/64 and so on. Table 1 shows the ratios used in the Pythagorean scale.

Table 1 – Ratios of each note to the lowest note used in the Pythagorean scale

Degree of scale	Interval	Ratio
1	Unison	1
2	Major 2nd	9/8
3	Major 3rd	81/64
4	Perfect 4th	4/3
5	Perfect 5th	3/2
6	Major 6th	27/16
7	Major 7th	243/128
8 = 1 - C	Octave	2/1

Table 1 gives the ratio of each note to the lowest note, the fundamental. In order to calculate the ratio between the notes, that is the intervals between them, we take the ratio of each note to the one preceding it. The result can be seen in Table 2.

94 This rests on the theory of numerical ratios presented in books 7–9 of Euclid's *Elements*.

Table 2 – Ratios of each note to the one preceding it in the Pythagorean scale

Degree of scale	Interval	Ratio
1		
2	tone	9/8
3	tone	9/8
4	semitone	256/243
5	tone	9/8
6	tone	9/8
7	tone	9/8
8	semitone	256/243

There is an elegant simplicity in the way that this use of simple ratios leads to the construction of the diatonic scale. This simplicity accords with the ideals of the philosophy promulgated by Pythagoras and many of his contemporaries which sought to explain the nature of the universe through numbers, ratios and proportions. Music theory too was mathematically based dealing with ratios, proportions and number relations. However, there are several flaws within this Pythagorean method of calculation: not all of the intervals of the scale can be produced accurately by strict Pythagorean methods; they do not satisfy the properties of the circle of 5ths (see Figure 1); and the intervals do not always match the acoustical properties found in the harmonic series.

Taking each of these problems in turn, Table 1 shows us that all five steps of a tone are in the ratio of $W = 9:8$ and the two semitones are in the ratio of $H = 256:243$. However, two half steps in this Pythagorean tuning do not add up to one whole step.[95]

$$H^2 = 256/243 \text{ x } 256/243 = 2^{`16}/3^{10} \neq 3^2/2^3 = W$$

95 Gareth Roberts. *From Music to Mathematics {Exploring the connections}.* (Baltimore: John Hopkins University Press, 2016): 121.

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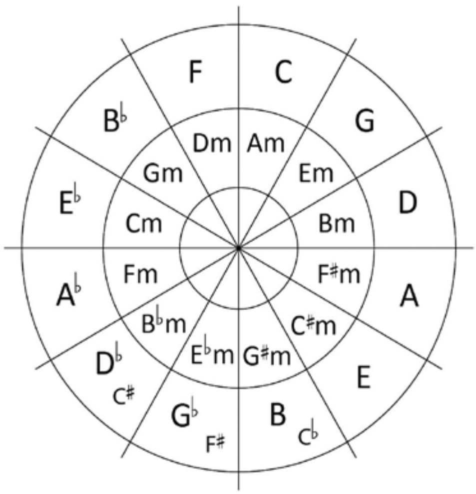
This discrepancy is referred to as the 'Pythagorean comma', defined as the gap between one whole step and two half steps. It can be found by dividing the ratio for a whole step by the ratio for two half steps.

$W/H^2 = 3^{12}/2^{19} \approx 1.013643265$

This means that raising the pitch by two half steps produces a slightly smaller interval than raising it by one whole step.

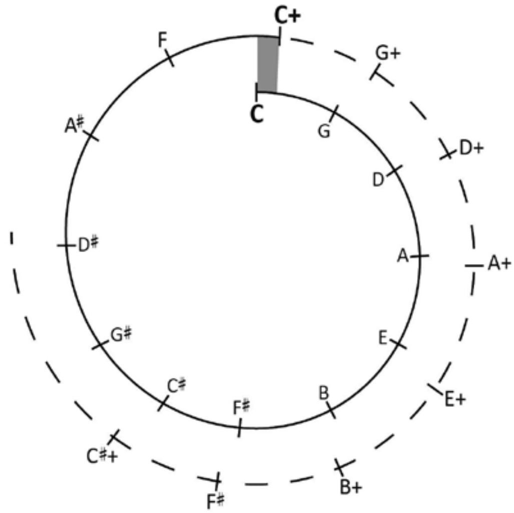
The next problem with the Pythagorean scale is related to the circle of 5ths. The circle of 5ths is represented by a circular diagram demonstrating the relationship between different keys (see Figure 1).

Circle of 5ths



It shows a series of chords whose roots are each a 5th higher than the previous chord e.g. C-G-D-A (see Figure 1). From any starting note the pitch is raised repeatedly until the starting point is returned to and the circle closes. This is the equivalent to raising the pitch by seven octaves. However, using the specified Pythagorean ratios, the span of seven octaves is not equivalent to twelve 5ths and there is a need to introduce a new note for each octave to fill the gap. This means that rather than returning to the starting point and closing the circle, the figure becomes an ever-increasing spiral (see Figure 2). The shaded area denotes the Pythagorean comma.

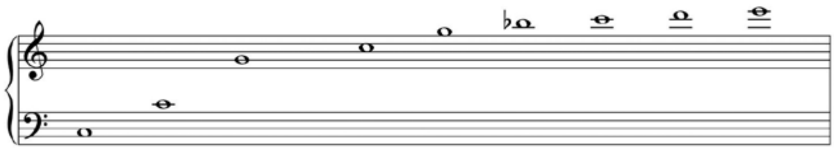
Spiral of 5ths



The harmonic series

Another problem lies in the correlation of the Pythagorean scale with the harmonic series. When a vibrating object, such as a string, is set in motion, it vibrates both as a whole, with a frequency called the fundamental (the lowest note or the first harmonic) and, with lesser intensity, in other sections as well. The harmonics (or overtones) generated may be represented in an ordered series called the harmonic series, a set of frequencies which are successive integer multiples of the fundamental (see Figure 3).

Figure 3 – The harmonic series



The frequencies are related by simple whole number ratios. In general, the *n*th harmonic of a series has a frequency which is *n* times the fundamental frequency. The harmonic series defines many of our intervals. As we have seen, the calculations used to make the Pythagorean scale were very limited and they missed out some of the most important ratios found in the harmonic series such as the major 3rd (5:4) and the minor 3rd (6:5). In this way the Pythagorean scale was out of sync with the laws of acoustics.[96]

Nevertheless, use of the Pythagorean scale persisted for hundreds of years. Another system of tuning known as equal temperament avoids the Pythagorean comma, but it was many years before it was in common use. The evolution of polyphonic music from the twelfth century onwards led to experimentation with alternative systems of tunings. Polyphony used several instruments or voices and music became more complicated harmonically; music was no longer largely restricted to the predominant use of three harmonic intervals - the octave, 4th and 5th – and 3rds and 6ths were gradually adopted.

Just intonation

Just intonation is based upon the first six notes of the harmonic series, the word 'just' used because this alignment is considered to be more acoustically pure or true. The ratios used are based on smaller numbers than those of the Pythagorean scale (see Table 3). The just major 3rd uses a ratio of 5:4, rather than 81:64, which is more consonant than its Pythagorean counterpart. Similarly the just major 6th is 5:3 (rather than 27:16) and the major 7th is 15:8 (rather than 243:128).

96 Eli Maor. *Music by the Numbers*. (Princeton: Princeton University Press, 2018): 19.

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Table 3 – Ratios used in just intonation

Degree of scale	Interval	Ratio
1	Unison	1
2	Major 2nd	9/8
3	Major 3rd	5/4
4	Perfect 4th	4/3
5	Perfect 5th	3/2
6	Major 6th	5/3
7	Major 7th	15/8
8 = 1	Octave	2/1

The idea of using the ratio of 5:4 was first suggested by two medieval British theorists, Theinred of Dover and Walter Odington and in ensuing decades there was much debate about different tunings by leading scientists and mathematicians across Europe, both the German mathematician Johannes Kepler (1571-1630) and the English mathematician Isaac Newton (1642-1727) devised scales. Kepler devised a scale derived from the ratios of planetary orbit minimum and maximum speeds and Newton devised a seven note diatonic scale based on the seven colour spectrum of the rainbow.[97] The French polymath Marin Mersenne (1588-1648) is credited with being the first to formulate rules governing vibrating strings, and the first to discern the nature of harmonics related to a fundamental note. In the search for a more consonant scale, the eighteenth century Swiss mathematician, Leonhard Euler, developed an elaborate mathematical theory which was based upon just-intonation ratios.

Although just intonation conforms more closely to the laws of acoustics than the Pythagorean scale, it is not without problems; the circle of 5ths is still not closed and there are two different ratios for the whole step within the scale. The ratio for the step between the first and 2nd degrees (a tone) is 9:8 whereas the step of a tone between the 2nd and 3rd degrees has a

97 Neil Bibby. *Music and Mathematics. From Pythagoras to Fractals.* (Oxford: Oxford University Press, 2003).

ratio of 10:9. The gap between these two different steps (9/8 x 10/9) is the ratio 81/80 which is known as the 'syntonic comma'.[98] Although this is not discernible to all listeners and does not pose a problem for instruments that have a continuous range of notes such as violins, for instruments with fixed keys or holes it makes playing in more than one key often difficult and sometimes impossible. Keyboard instruments were particularly problematic in this respect. The celebrated German/Viennese piano maker Johann Jakob Könnicke (1756-1811) attempted to solve this problem when in 1796 he invented the 'Harmonie-Hammerflügel', a keyboard instrument with six diatonic manuals that divided the normal 12-key octave into 31 notes to enable a purer tuning. Unfortunately it was difficult to build, as well as difficult to keep in tune and to perform on.[99]

Equal temperament

The answer to this problem was to be found in equal temperament tuning. The word tempered refers to the fact that just intonation has been adjusted or compromised. It is so named because the scale is divided into 12 equal semitones. In this system, the circle of 5ths is closed. This means that there are no discrepancies between different keys, all the intervals across keys are equivalent, making it possible to play music with multiple key changes and chromatic harmony.

Rather than using the whole number ratios found in the Pythagorean scale and just intonation, the Flemish mathematician Simon Stevin (1548-1620) came up with the idea of letting the half step (semitone) equal $\sqrt[12]{2}$, the 12th root of 2 which can also be written as $2^{2/12}$. In other words, when multiplied by itself 12 times, $2^{2/12} = 2$. This is an irrational number in contrast with the rational numbers used in the aforementioned systems. The Pythagoreans were strong believers in the importance of rational numbers, those numbers which are either integers or can be written as a ratio (or quotient) of two integers. An irrational number is a real number which cannot be expressed as

98 Roberts, *From Music to Mathematics*, 129.

99 Michael Latcham. 'Könnicke, Johann Jakob' in Grove Music online.

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an integer or as a quotient of two integers. Irrational numbers have infinite, non-repeating decimals.

Having established $2^{2/12}$ as the interval for a half step, Stevin went on to define all the other intervals by multipliers of this step (see Table 4). The numbers in the 3rd column are easily found by calculating the number of half steps in each interval.

Table 4 – Ratios used in Stevin's equal temperament

Degree of scale	Interval	Ratio
1- -C	Unison	1
2 - D	Major 2nd	$2^{2/12}$
3 - E	Major 3rd	$2^{4/12}$
4 - F	Perfect 4th	$2^{5/12}$
5 - G	Perfect 5th	$2^{7/12}$
6 - A	Major 6th	$2^{9/12}$
7 - B	Major 7th	$2^{11/12}$
8 = 1 - C	Octave	2

Marin Mersenne (1588-1648) made an important contribution to the theory of tuning and was an advocate of equal temperament. He also reassessed the nature of consonance and dissonance. The most famous early work to use the system and exploit all 24 keys is J S Bach's *Well-Tempered Clavier* – the two books were written in 1722 and 1738-1744 each comprising a prelude and fugue in each major and minor key. It took a long time for equal temperament to be adopted across Western music; as late as 1851, not one of the British organs at the Great Exhibition was equally tempered.[100] Equal temperament is now widely thought of as the normal tuning of the Western 12-note chromatic scale. Table 5 compares the frequencies of notes in a scale of A major according to the ratios of the Pythagorean scale and the equal tempered scale measured in Hertz.

100 Bibby, Music and Mathematics, 27.

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Frequency expresses the number of repetitions during a certain length of time. It is measured in time units, usually seconds, and called Hertz after the German physicist Heinrich Hertz (1857–1894). A tuning fork that vibrates 440 times back and forth in one second, concert A, has a frequency at 440 Hz.[101] The set of frequencies 100, 200, 300, 400, 500 Hz … is a harmonic series whose fundamental is 100 Hz and whose 5th harmonic is 500 Hz.

Table 5 – A comparison of frequencies used in equal temperament and the Pythagorean scale

	A	B	C#	D	E	F#	G#	A
Equal temperament	220.0	246.9	277.2	293.7	329.6	370.0	415.3	440.0
Pythagorean scale	220.0	247.5	278.4	293.3	330.0	371.3	417.7	440.0

Table 6 shows the approximate differences in frequencies for a whole step (tone) and half step (semitone) used in the equal tempered scale and the Pythagorean scale.

Table 6 – A comparison of the frequencies for a tone and a semitone used in equal temperament and the Pythagorean scale

	tone	**semitone**
Equal temperament	$2^{1/12} = 1.122$	$2^{2/12} = 1.0595$
Pythagorean	$9/8 = 1.125$	$256/243 = 1.0535$

Another commonly used way of measuring the intervals in different tuning systems is through the use of cents. Cents are based on a logarithmic scale and there are 100 cents in an octave. The system was introduced in the late nineteenth century by the English mathematician Alexander Ellis (1804-90). In his analysis of the scales used in various European musical traditions, he showed that the diversity of different systems could not be explained by a single physical law.

101 Concert A is the standard tuning note in the UK and USA. Until the nineteenth century musical pitch was not standardized and the levels varied widely across Europe. Nowadays ensembles which specialise in music of the Baroque period have agreed on a standard of A = 415 Hz.

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Table 7 gives a comparison of the three tuning systems discussed so far measured in cents. It can be seen that although equal temperament is a close approximation to the perfect 5th is significantly sharper than a major 3rd in equal temperament.

Table 7 – A comparison of the frequencies used in the Pythagorean scale, equal temperament and just intonation measured in cents

Scale degree	**Interval**	**Pythagorean scale**	**Equal temperament**	**Just intonation**
1	Unison	0	0	0
2	Major 2nd	203.9	203.9	200
3	Major 3rd	407.8	386.3	400
4	Perfect 4th	498.0	498.0	500
5	Perfect 5th	702.0	702.0	700
6	Major 6th	905.9	884.4	900
7	Major 7th	1109.8	1088.3	1100
8 = 1	Octave	1200	1200	1200

Moving away from the seven-note diatonic major scale with its use of tones and semitones, the next chapter goes on to look at a variety of other scales using a different number of notes and often including the interval of a microtone.

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