12 Fractals and chaos theory

FRACTALS BELONG TO a class of curves or complex geometric shapes in which each part has the same statistical character as the whole, that is, it is made up of smaller scale copies of itself. Their properties are found in many irregularly shaped objects and non-uniform phenomena in nature such as coastlines and mountain ranges. The word fractal is derived from the Latin word *fractus* meaning broken.

Can fractals be found in music? This chapter examines the evidence. It opens by looking at fractals in mathematical terms describing the characteristics and properties of the von Koch curve and the Cantor set and outlining the early work of Benoit Mandelbrot in this field. Over the years various attempts have been made to identify fractal patterns in music, and several ideas have been proposed including the use of nested sequences and self-similarity. These include the prolation canons dating from the fifteenth century by composers such as Ockeghem and des Prez. More recently the Hungarian composer György Ligeti showed a great interest in both fractals and chaos theory, inspired by the computer-generated illustrations of Heinz-Otto Peitgen and Peter Richter. Several compositions of Ligeti are examined and analysed in terms of their fractals characteristics. The American composer Charles Wuorinen was also fascinated by fractal geometry and he used some of its mathematical properties in the construction of his music. In recent years several composers have taken advantage of the capabilities of computers to generate fractal inspired pieces, amongst these the Norwegian composer Rolf Wallin. Robert Sherlaw Johnson was one of the first to explore this field and he provides an illuminating critique of his method. Nowadays there are increasingly sophisticated software packages specifically designed

10.5920/mathsandmusic.12

to created fractal music. The chapter concludes with a brief discussion of the original question 'Can fractals be found in music?'

What is a fractal?

Fractals commonly have 'fractional dimension' a concept which was first introduced by the mathematician Felix Hausdorff in 1918 and a measure of their complexity. Although fractional dimension, and other key concepts associated with fractals, had been known about for some time, it was not until the French-American polymath Benoit Mandelbrot (1924-2010) coined the term 'fractal' in his 1975 book on the subject that more interest was shown in the field.[217] Later Mandelbrot's seminal 1982 book, *The Fractal Geometry of Nature* catalogued the ubiquity of geometric patterns found in nature and is widely credited for bringing fractals to popular attention. He highlighted the need for fractal mathematics and pointed out that they could be useful in applied mathematics for modelling a variety of phenomena from physical objects to the behaviour of the stock markets. Since the 1980s virtually every scientific field has been explored from a fractal viewpoint and fractal geometry has become a major area of mathematics. Fractals are useful in modelling structures in which similar patterns recur at progressively smaller scales, as well as describing partly random or chaotic phenomena such as crystal growth and galaxy formation.

The von Koch curve

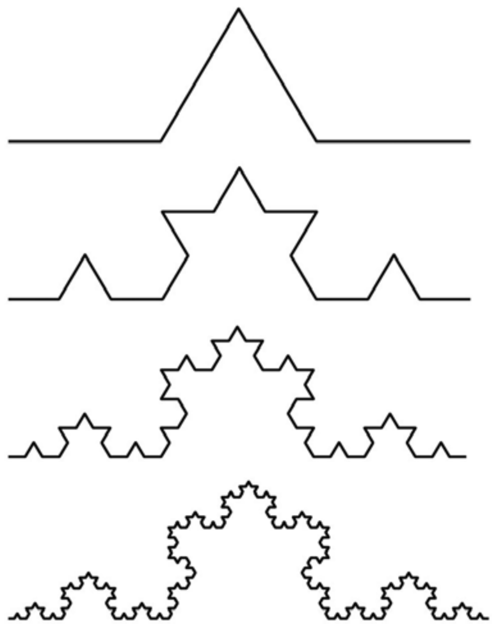
The properties of fractals can be illustrated in the von Koch curve (see Figure 1). An intricate and complicated object, the main von Koch curve contains many tiny von Koch curves which are made up of smaller scale copies of itself. This property is known as self-similarity. A self-similar object is one whose component parts resemble the whole. Details or patterns are reiterated at progressively smaller scales so that in effect, it remains invariant under

217 Benoit Mandelbrot. *Les Objets Fractals: Forme, Hasard et Dimension*(1975) later translated in 1977 as *Fractals: Form, Chance and Dimensio*. London: W H Freeman and Co,1975).

10.5920/mathsandmusic.12

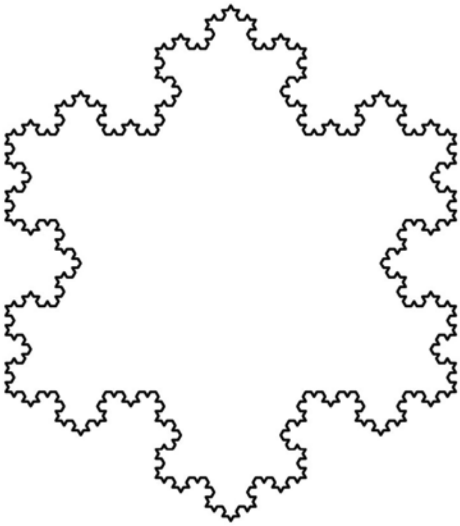
changes of scale, that is, it has scaling symmetry. In classical mathematics curves are usually described in terms of tangents but the von Koch curve does not have a well-defined slope or direction at any one point, hence it is too irregular to be described in traditional mathematical language. Mathematicians disagree on a precise definition, but a fractal is typically described as exhibiting self-similarity,

The von Koch curve



The curve is constructed by repeatedly replacing the middle third of each line segment with the other two sides of an equilateral triangle. Because the same step is repeated over and over again this is known as a recursive construction. When three von Koch curves are fitted together they form a snowflake curve (see below).

The von Koch snowflake



The main tool of fractal geometry is fractional dimension. The von Koch curve has dimension

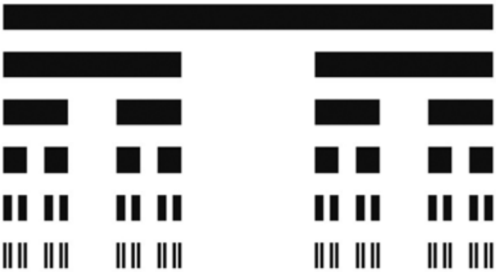
log 4/log 3 = 1.262.[218]

The Cantor set

In 1883 the German mathematician George Cantor (1845-1918) introduced what he referred to as the middle third Cantor set. This was obtained by repeatedly removing the middle thirds of intervals which results in a basic self-similar fractal made up of two scaled ½ copies of itself (see Figure 3).

The middle third Cantor set

218 Kenneth Falconer. *Fractal Geometry. Mathematical foundations and Applications*. (Chichester: Wiley, 2003): xix.



Each stage of the construction is obtained by removing the middle third of the previous stage. Take the closed interval [0, 1]. First delete the open interval that forms the middle third [1/3, 2/3] then delete [1/9, 2/9], then delete [7/9, 8/9] and so on.

In 1918 Felix Hausdorff showed that the middle third Cantor set had dimension of $\log 2/\log 3=0.631$.

In summary the main features of fractals (F) as given by Kenneth Falconer in his classic text *Fractal Geometry. Mathematical Foundations and Applications* are:

- F has a fine structure i.e. detail on arbitrarily small scales
- F is too irregular to be described in traditional geometrical language
- Often F has some form of self-similarity, perhaps approximate or statistical, made up of small scale copies of itself in some way
- In most cases F has a simple recursive construction.[219]

Early attempts to find fractals in music

Although there is no widely accepted consensus as to what constitutes a fractal pattern in music, a variety of statistical tools have been proposed over the years. The earliest contributions came from the American physicists Richard F. Voss and John Clarke in the 1970s who analysed the spectral density of audio power (effectively loudness) in a range of recordings from news radio, to Scott Joplin rags and Bach's Brandenburg Concerto No. 1.

219 Falconer, *Fractal Geometry*, xxv.

10.5920/mathsandmusic.12

Through this they identified what they referred to as scaling behaviours (the word fractal had not yet been coined) in the fluctuations of volume as well as some pop music melodies.[220] They observed that both pitch and volume fluctuations showed 1/*f* distribution, a type of distribution often associated with the hierarchical structure of fractals.[221]

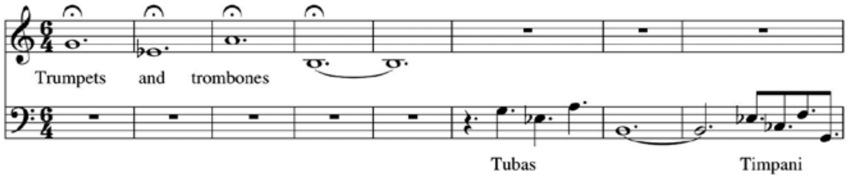
Twenty years later, Henderson-Sellers and Cooper proposed a definition of musical self-similarity based on note lengths analogous to the Cantor set. They specified how the faster passages are to be related to the slower ones arriving at a definition of a musical fractal based on the development of a melodic line; different time scales appear sequentially, in a single line of music, so that the entire passage is built from copies of the original idea written in shorter notes. Each faster copy is transposed in such a way that it begins with a note from the previous order so that every new order retains the memory of all previous orders, creating a self-similar structure. The melody is thus constructed through layers of nested sequences in a single line of music, which becomes faster and faster at each order, but retains the outline of the slower version in the leading notes of each scaled copy.[222] This algorithm is clearly quite mechanical and prescriptive; it cannot be sustained musically over any length of time and examples found in music depend on creative adaptations. Such an example can be found in the opening bars of 'Uranus, the Magician' from Holst's *The Planets* (1917). Here the opening statement is made by trumpets and trombones in dotted semibreves, this is followed by a second statement by tubas, but this time the note values are halved. The note values are halved again for the last statement by the timpani. As can be seen in this final statement the pitch order of the notes is no longer followed strictly.

220 Voss RF, Clarke J. 1/f noise in music and speech. Nature 1975;258:317–8 and Voss R F, Clarke J.. J Acoust Soc Am 1978; 63(1). 258-163.

221 In terms of noise 1/f noise (or fractal noise) is a signal with a frequency spectrum such that the power per frequency interval is inversely proportional to the frequency of the signal.

222 Henderson-Sellers B, Cooper D. 'Has classical music a fractal nature?: a reanalysis. Comput Hum 1994, 27(4): 277–84.

10.5920/mathsandmusic.12



The American musician and mathematician Harlan Brothers has been exploring fractals in music for some years and has written extensively on the topic. He has pointed out that identifying fractals in music is more problematic than seeing them in an image. 'Unlike a picture, which is all laid out so that you can instantly see the structure, music is fundamentally a serial phenomenon… With music, the whole piece takes shape in your mind. This makes it more challenging to identify the self-similarity'.[223] It may be difficult to perceive fractals in music without looking at the score or listening to the piece repeatedly.

In 2007 Brothers published a study of the fractal structure of Bach's *Cello Suite No. 3* where he found that, at various points, patterns of long and short notes within bars reappeared as patterns of long and short phrases at larger scales. He referred to this as 'structural scaling' noting that this self-similarity bore a striking resemblance to the Cantor set. According to Brothers, Mandelbrot had suspected that music contained fractal patterns but that he didn't have time to investigate further. Subsequently Brothers contributed a chapter to *Benoit Mandelbrot: A Life in Many Dimensions* in which he lists the seven types of statistical self-similarity that he had found in music: duration scaling, the distribution of note lengths; melodic interval scaling, the distribution of changes in pitch; melodic moment scaling, the distribution of the changes in melodic intervals; harmonic interval scaling; structural scaling; and motivic scaling. Motivic scaling involves the simultaneous rendition of the same motif or theme at various tempos resulting in an intricate interplay of the theme with its faster or slower versions. An example of this musical self-similarity is the prolation canon (see pages 150-1) where each voice in the canon sings or plays the same music, but at different speeds with voices entering either successively or simultaneously.

223 Harland Brothers. 'Structural scaling in Bach's Cello Suite No. 3'. *Fractals. 15, 2007: 90.*

The prolation canon

The fifteenth-century Franco-Flemish composer Johannes Ockeghem was the master of the prolation canon and possibly its inventor.[224] His Missa prolationum (c. 1450) presents a series of prolation canons each offering a different variant of the form and whose interval of imitation moves from the unison progressively through to the octave. Many of his compositions are grounded in some rational, often ingenious, conception, some also serving a didactic as well as musical purpose. 'Kyrie Eleison I' is a double prolation canon, where in the opening bars, two separate pairs of voices, soprano and bass, alto and tenor use two different motifs. The four parts begin together, each of the four voices follow the entire six-bar theme exactly once. For each pair, the ratio of tempos is 3:2. This is clearest to see in the alto and tenor parts where the alto uses four minims and a crotchet (all divisible by 2), whereas the tenor part uses four dotted minims and a dotted crotchet (all divisible by 3). After this passage, the music continues as a regular canon (see page 150).

Prolation canons continued to be composed in the twentieth century in, for example, Arvo Pärt's vocal work *Cantus in Memoriam Benjamin Britten* (1977) and the first movement of Shostakovitch*'s Symphony No. 15 (1971). Cantus in Memoriam Benjamin Britten*, scored for string orchestra and bell, is a prolation canon in five voices, with a ratio of speeds of 16: 8: 4: 2: 1 (violin I, violin II, viola, cello, and bass). The theme is based on the descending octave of an A minor scale, developed sequentially with one note added at each repetition. Thus, the theme proceeds as A-A-G-A-G-F-A-G-F-E-A thus creating what could be described as an ever-elongating fractal sequence. In the same way as Ockeghem and des Prez, the composers were using this compositional device consciously, albeit being unaware of the concept of fractals. Once the concept of fractals became well-known, a number of composers used fractal mathematics as part of the compositional process. Amongst these are the Hungarian György Ligeti, the Norwegian composer Rolf Wallin and the American Charles Wuorinen.

224 John McDonough and Andrzej Herczynski. *Chaos, Solitons and Fractals* (Science Direct 170, 2023).

10.5920/mathsandmusic.12

The use of fractals by György Ligeti

György Ligeti had a keen interest in fractals: he was well-informed and knowledgeable, and he took some of the ideas he had learnt and applied these to his compositions. In a 1990 interview, Ligeti commented on the influence of mathematics and fractals in his compositional process.[225] He said, '… in my music, there are mathematical considerations of fractal geometry. Its kind of growth and its processes of pattern transformation have occupied me for a great deal'.

In 1985 Ligeti first met the German mathematician Heinz-Otto Peitgen (b 1945). Peitgen was a pioneer of fractal geometry who had helped to introduce fractals to the broader public, partly through the publication of his lavishly illustrated book of computer-generated images *The Beauty of Fractals*[226]. Their friendship lasted until Ligeti's death in 2006. In Peitgen's words 'He introduced me to new music and to him I opened the door to the study of fractals, fractal geometry and other fields of mathematics … However dissimilar Ligeti's origin and life as a musician was from my scientific background, we both saw that what we did was, in some way, remarkably connected and interwoven'.[227]

It is important to note that Ligeti did not apply fractal geometry to his music with systematic mathematical rigour; rather he took some of the principles and formulated compositional procedures based loosely upon them. He once said that only two of his compositions were 'deliberately based on ideas from contemporary mathematics, the first piano etude 'Désordre' (1985) which is self-similar - an iterated structure based consciously on the Koch snowflake' - and the fourth movement of the *Piano Concerto* - 'a fractal piece'.[228]

---

225 Stephen Satory. 'Colloquy: An interview with Gyorgy Ligeti'. *Canadian University Music Review*. 1990.

226 Heinz-Otto Peitgen & Peter Richter. *The beauty of fractals* (Berlin & New York, 1986).

227 Heinz Otto Peitgen. 'Continuum, Chaos and Metronomes – A Fractal Friendship' in Louise Duchesneau and Wolfgang Marx. *György Ligeti. Of Foreign Lands and Strange Sounds*. (Martlesham, Suffolk: Boydell Press (2011).

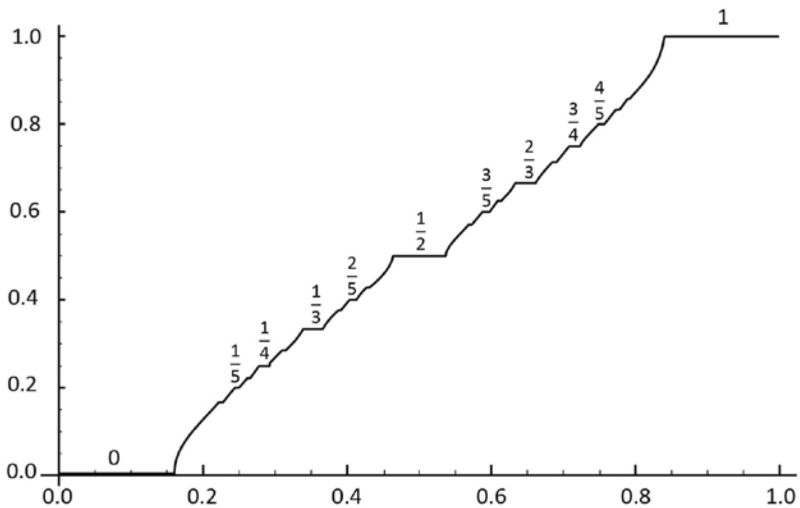
228 Ligeti in conversation with Heinz-Otto Peitgen and Richard Steinitz, Huddersfield Contemporary Music Festival (November 1993) cited in Richard Steinitz. 'The Dynamics of Disorder'. *The Musical Times,* Vol. 137, No. 1839 (May, 1996): 8.

10.5920/mathsandmusic.12

Ligeti – 'L'escalier du diable (The Devil's Staircase)'

These two pieces will be looked at later in more detail, but first some mention should be made of another piano etude - 'L'escalier du diable' (The Devil's Staircase)' (1995-2001), the thirteenth etude which makes direct reference to a construction based on the Cantor set. The devil's staircase, sometimes referred to as the infinite staircase or the Cantor function, has unequal ascending steps constructed by using, for example the recursive 1/3 to 2/3 proportions of the most common middle-thirds Cantor set (see below).

The Devil's staircase



'L'escalier du diable' is a flamboyant virtuosic piece, *presto* with polyrhythms and rising chromatic scales getting progressively higher. Richard Steinitz describes it thus:

> The rotating repetitions of the principal motif climb up an endless staircase, flight upon flight, octave by octave. Meanwhile, the pattern restarts in low register, as if emerging from dark subterranean dungeons. Like the synthetic spiral - the computer-generated illusion

> - it seems to rise ad infinitum, yet remains enigmatically the same. Its patterns can be represented numerologically, but are intended first and foremost to generate polyrhythmic energy. Here we have Ligeti at his most melodramatic, in music overwhelmingly powerful and thrilling.[229]

Although it has recursive qualities, it does not follow the strict mathematical formula prescribed by the devil's staircase. As Steinitz writes, 'Ligeti constructs his musical staircase using his own numerical system', and although 'Its patterns can be represented numerologically' they 'are intended first and foremost to generate polyrhythmic energy'.[230]

Ligeti – 'Désordre'

'Désordre' is the first of Ligeti's piano etudes and, as was mentioned above, Ligeti described it as having an 'iterated structure based consciously on the Koch snowflake'. The piece is marked *molto vivace*, very fast, and could be described as a study in rapid polyrhythms moving up and down the keyboard. The right hand plays only white keys and the left hand is restricted to the black keys in a pentatonic mode. Although both the right hand and left hand have eight quavers to the bar, the barlines and accents are placed differently. Hence the material of the two hands unfolds simultaneously but independently. As Steinitz puts it, 'At first glance their musical material looks alike. However, it is their dissimilarity which is crucial.' Later in the piece the same phrase structure is resized, the melodic material is similar but the phrase lengths are shorter and the accents appear more frequently.

In the words of Steinitz, 'Repeating an operation over and over again, on ever smaller scales culminates almost inescapably in a self-similar structure, a classic example being the 'Koch curve''. How do these musical procedures measure up against Falconer's fractal characteristics that were listed earlier? The music has elements of a 'fine structure' with small scale detail and it exhibits some form of 'self-similarity'. However, although it could be argued

229 Richard Steinitz. 'Maths and Chaos'. *The Musical Times, Vol. 137, No. 1837 (March, 1996): 19.*
230 Ibid.

10.5920/mathsandmusic.12

that it has a 'recursive construction', this is not created in an orderly fashion in the way that a von Koch curve would be by repeating the same steps and more specifically replacing middle thirds.

Ligeti – The Piano Concerto

Ligeti was particularly influenced and inspired by computer fractal images particularly those in *The Beauty of Fractals. Indeed* he was so fond of one of the images of a Julia set that Peitgen dedicated it to him naming it the Ligeti fractal, and Peitgen's '29 Arms at Seahorse Valley', a mathematical image of the Mandelbrot set, was an inspiration for the *Piano Concerto.*[231] Although Ligeti understood the mathematical calculations behind fractals, these were not directly involved in his compositional methods. When describing the fourth movement of his *Piano Concerto,* he wrote

> The whole structure of the piece is self-similar and the impression we get is that of a giant interconnected web … This vast self-similar maelstrom can be traced back - indirectly – to musical ideas which had been unleashed by beautiful fractal imaged of the Julia and Mandelbrot sets … Not that I use iterative calculations when I composed this piece … rather, it originated out of an intuitive synthetic correspondence on a poetic level.[232]

Steinitz describes the movement thus,

> Indeed, we do find in this movement a particular musical gesture repeated endlessly at different magnifications... The movement is literally flooded with iterations of a three-note fragment, its extensions, contractions and derivatives - all progressively superimposed in varying tempos, with varying articulation, coloration and magnitude. The musical design builds up like a complex graphic image scanned,

231 Richard Steinitz. *György Ligeti. Music of the Imagination.* (London: Faber and Faber, 2003).

232 Gyorgi Ligeti 'Zu meinem Klavierknonzert (1988)' in Gesammelte Schriften, Vol. 2 :299, translated by Louise Duchesneau cited in *György Ligeti. Of Foreign Lands and Strange Sounds.* Boydell Press (2011): 103.

10.5920/mathsandmusic.12

> pixel by pixel, on a computer screen. By the end of the movement we've arrived at an elaborate, multi-dimensional fantasy of self-replicated whirls, riotous colour and polymetric energy, an exuberant vortex of order and chaos, symmetry and freedom, like the magnificent fractal images of Mandelbrot and Peitgen.[233]

We are reminded of the problems in identifying fractals in music where, as Brothers pointed out, 'Unlike a picture, which is all laid out so that you can instantly see the structure … With music, the whole piece takes shape in your mind'.[234] If, as Hans Lauwerier writes in the introduction to *Fractals. Images of Chaos*, 'A fractal is a geometric figure in which an identical motif repeats itself on an ever-diminishing scale' then this movement of the *Piano Concerto* movement cannot be described as fractal.235 It would not be true to say that it unfolds progressively over time moving systematically from large statements of the three-note fragment to ever smaller more intricate patterns although some aspects of this are evident, the ever-decreasing values are not systematic. The movement opens with loud statements of the three-note motif interspersed with bars of silences. Over the course of the movement the music does become more complex but reaching a climax about three quarters of the way through, rather than the end, before petering out to a pianissimo conclusion. Perhaps the fractal images are more evident in the dynamics and orchestration where the finer detail is represented by the high notes of the glockenspiel and piccolo, and towards the end of the movement the piano makes increasing use of the higher register, moving upwards, higher and higher to the very top of the keyboard register.

Tom Johnson and Naryana's Cows

Much of the music of the American composer Tom Johnson (b. 1939) follows mathematical principles and logical procedures such as permutations,

---

233 Richard Steinitz. 'Weeping and Wailing'. *The Musical Times,* Vol. 137, No. 1842 (Aug, 1996): 17-22.

234 Brothers. 'Structural scaling', 90.

235 Hans Lauwerier. *Fractals. Images of Chaos.* (London: Penguin. 1991): xix.

combinations, tiling, block design and fractals.[236] He too was inspired by Benoit Mandelbrot's book *Fractals* (1977) with its 'pictures of the Koch curve, the Minkowsky sausage, the Sierpinski sponge, the Peano island and the Cantor triadic bar'. For Johnson they were 'essentially maps showing the route that I was about to follow'.[237]

When interviewed by New Music USA in 2003, Johnson was asked 'What role has theory played in your compositions and how important is it for people to know the theory behind the music in order to appreciate it?' He replied by saying 'Theory is when you know everything about it but it won't work. Practice is when it works but you don't know why' adding that good composers not only compose but also think a lot about how they compose. This was the thinking behind what he describes as his 'purely technical' book, *Self-Similar Melodie*s [238](1996), a didactic study with specially composed examples based on this characteristic of fractals.[239]

Many of his pieces use a variety of techniques in self-similarity, although as he pointed out in his IRCAM lecture of 2006, 'true self-similar structure is relatively rare.' In the lecture he defines these pieces as 'music that somehow contains itself within itself and does so on at least three different levels of time. The use of self-similarity is quite evident to the listener his piece, *Narayana's Cows*. It was 'derived from a 14th-century Indian problem in which a mother cow gives birth to a female calf at the beginning of each year and each calf becomes a mother and does the same thing beginning in her fourth year. By the 10th year, the herd numbers 60 mothers and daughters… repeating one process over and over always results in some kind of self-similarity' The following illustration shows the 60-note Narayan sequence that has evolved by the 10th year. The first line of numbers below represents the 60 cows of the herd in its 10th year. The second line singles out the 19 mothers, that is, the quarter-notes, of the 10th year, and this happens to be the notes of the 19

236 For more information about Johnson's use of tiling procedures see - Tom Johnson. 'Tiling in my music'. https://web.northeastern.edu/seigen/MusicDIR/Tiling_in_my_music_Johnson.pdf

237 Tom Johnson. 'Self-Similar Structures in my Music: an Inventory lecture' presented in the MaMuX seminar, IRCAM, Paris, Oct. 14, 2006 in context with a lecture on self-similarity by mathematician Emmanuel Amiot

238 Tom Johnson. *Self-Similar Melodie*s. (Two Eighteen Pr, 1996)

239 https://newmusicusa.org/nmbx/What-role-has-theory-played-in-your-compositions-and-how-important-is-it-for-people-to-know-the-theory-behind-the-music-in-order-to-appreciate-it-Tom-Johnson/ April 2003

cows of the 7th year, and the mothers of the 7th year are equivalent to the six mothers and daughters of the 4th year, shown on the last line.

1 2 2 2 2 3 2 3 3 2 3 3 3 2 3 3 3 3 4 2 3 3 3 3 4 3 4 4 2 3 3 3 3 4 3 4 4 3 4 4 4 2 3 3 3 3 4 3 4 4 3 4 4 4 3 4 4 4 4 5

1 2 2 2 2 3 2 3 3 2 3 3 3 2 3 3 3 3 4

1 2 2 2 2 3

In Johnson's musical work *Narayana's Cows* there are two layers: one verbal and the other musical. In the verbal layer, the speaker transposes Narayan's numbers into words: 'The first year there is only the original cow and her first calf. The second year there is the original cow and two calves. The third year there is the original cow and three calves. The fourth year the oldest calf becomes a mother, and we begin a third generation of Narayana's cows.' And so on until the seventeenth year is reached. The second layer is made up of musical sounds where the pitch correlates with the cows' generation, and the rhythm correlates with the difference between a cow (long note) and a calf (short note). Clearly unfolding to the listener, together they evolve into an accumulative and complex melody.

Chaos theory

In the words of Peitgen, 'Ligeti was just as fascinated by the discovery of deterministic chaos as he was by Benoit Mandelbrot's fractal geometry'. Chaos theory (the science of dynamical systems) focuses on the behaviour occurring in a system under iteration and is the study of apparently random or unpredictable behaviour in systems governed by deterministic laws. Fractals form part of the visual identity of chaos. In recent decades a diversity of systems has been studied that behave unpredictably, despite the fact that the forces involved are governed by well-understood physical laws. The common element is sensitive dependence to initial conditions and to the way in which they are set in motion. The connection between chaos and fractals is the strange attractor. In the field of dynamical systems an attractor is a set of states towards which a system tends to evolve from a wide variety of starting conditions. A strange attractor is a complicated set with

10.5920/mathsandmusic.12

a fractal structure. One effect of chaos is commonly known as the Butterfly Effect, where a small change in initial conditions can lead to a large change in the behaviour of a system. The term comes from the meteorologist Edward Lorenz who accidentally discovered the effect in 1961 while trying to model the weather. In one run-through he entered 0.506 into his data instead of 0.506127 and was surprised to see that the original results were vastly altered by such a small change in one variable; where there was originally sun, there was now rain, a windy day became a calm day and so on. The Butterfly Effect therefore describes how a small change in one state of a deterministic system can bring about large differences in a later state with the potential to render long-term predictions impossible.

According to Steinitz 'Ligeti realised that the new theories which sought to explain the precarious balance between order and disorder, pattern and chaos, and the apparent origin of both conditions in measurable deterministic situations, had intriguing parallels with the way he composed.'[240] In a later article he writes 'That he [Ligeti] now views this interaction from the vantage-point of current mathematical thinking is indicated by his naming the first study after a crucial issue in the science of dynamical systems (more commonly known as chaos theory), the concept of 'disorder'.[241] *'Désordre'*, writes Steinitz 'demonstrates how tiny discrepancies quickly breed confusion. Albeit in microcosm and in a finite context, Ligeti illustrates a fundamental idea of chaos - that small differences in initial conditions rapidly lead to dramatic outcomes.' In his 2012 lecture on 'Chaos and Fractals in Music' the mathematician Gareth Roberts explained how this is achieved:

> Each hand opens with identical 8-beat rhythmic patterns (3 + 5). In the fourth measure, the right hand drops a beat, playing a 7-beat pattern rather than an 8-beat one, but continues the 8-beat pattern for the next three measures. This small change starts to cause a big shift, audible for the listener due to the shifting accents in each hand. In the eighth measure, the right hand drops another beat, playing 7 instead of 8. Now, the left hand is two beats ahead instead of one. Again, the right hand only drops a beat in this one measure. The "iterative"

240 Steinitz, 'Maths and Chaos', 15.

241 Steinitz. 'Dynamics of Disorder', 8.

> process of dropping a single beat continues, as the right hand drops a beat approximately every four measures so that the two hands become completely out of synch, and the butterfly effect is realized.[242]

Closer analysis of '*Désordre*' shows that this system is not rigorous, however, so it is not obeying the deterministic laws of chaos theory. In the same lecture, Roberts argues that the American composer Steve Reich employs a similar Butterfly Effect device where 'small changes in rhythmic structure (e.g. slight phase shifts) lead to big changes in the music' in, for example, *Clapping Music* (see pages 32-4). In *Clapping Music* (for two players) it could be argued that Reich applies deterministic laws in the way that the phase shifts he uses are applied systematically. In phase shifting constantly repeated patterns are subjected to gradual changes, one part repeats constantly and another gradually shifts out of phase with it, a compositional device that can be seen in terms of a mathematical translation. In *Clapping Music* the phase shifting is systematically applied quaver by quaver, with Player 2 shifting further out of phase with Player 1, effect is one of huge rhythmic diversity until eventually the parts come together again. It is interesting to note that *Clapping Music* was composed in 1972 so it pre-dates the period when chaos theory was widely known.

The use of fractals and chaos theory in the music of Rolf Wallin

The Norwegian composer Rolf Wallin (b. 1957) combines computer-generated systems and fractals with his intuitive approach to composition. He is dismissive of what he refers to as Ligeti's 'supposedly 'fractal' music'. In a lecture given in 1989 he said

> … when one investigates Ligeti's supposedly 'fractal' music, for instance his piano etudes from 1985, one finds that his music could very well be achieved without having heard of the existence of chaos theory or

242 Gareth Roberts. 'Chaos and Fractals in Music'. Maths and Music: Aesthetic Links, Montserrat, April 16, 2012.

> having seen the Peitgen-Richter pictures. The techniques he uses are derived exclusively from traditional compositional practice, and the music is neither more nor less fractal than African drum rhythms or a Bartok string quartet.[243]

Wallin outlined a more mathematical approach to composing with fractals in a lecture at the Nordic Symposium for Computer Assisted Composition, Stockholm in 1989. Wallin describing several ways to apply fractal mathematics and chaos theory to music using different mathematical formulas. As he points out there are 'countless ways of applying fractals to music'. His rationale for the use of computers and chaos theory is that he can give the computer a 'task where the direction is clearly defined, but where the details are unpredictable, and therefore generating a balance between global order and local disorder'. As he says, 'this has kicked me out of some of the tracks I have been stuck in'.

Stonewave (1990), one of his most popular and frequently performed percussion pieces, is made purely from fractal mathematics and in his programme note to *Stonewave* he writes that 'These formulas, used in the fast growing field of "Chaos theory", are relatively simple, but they generate fascinating and surprisingly "organic" patterns when shown graphically on a computer screen, or played as music.' He adds that 'One should think that such a mathematical approach would lead to sterile and 'theoretical' music. The sound world of *Stonewave*, however, is not one you would associate with math books. The steady, insistent pulse, and the use of sequences put squarely up against each other or divided by long rests suggest an invisible ritual.'

The use of fractals in the music of Charles Wuorinen

Much of the catalogue of the American composer Charles Wuorinen (1938-2020) was constructed using idea taken from fractals; the generation of infinite and chaotic structures through the iteration of finite elements. He was fascinated by fractal geometry, in particular the work of Benoit Mandelbrot

243 Rolf Wallin. Lecture given at the Nordic Symposium for computer Assisted Composition, Stockholm, 1989.

10.5920/mathsandmusic.12

whose ideas seemed to confirm his own ideas about musical structure and form. Mandelbrot had an interest in Wuorinen's music and, just as Ligeti struck up a long friendship with the mathematician Peitgen, so too did Wuorinen when he got know Mandelbrot. In his essay, 'Music and Fractals', Wuorinen identified the ways in which fractals arose in his music – acoustically, rhythmically, structurally and in terms of pitch generation. Acoustically, the signal has a 1/*f* distribution (as identified by Voss and Clarke). Rhythmically, his music has a self-similar, hierarchical division of time, and pitch generation is similar across different levels. In terms of structure, a single harmonic progression can be used to determine a single phrase, part of a section, or a whole section.[244] In this way Wuorinen developed a logical structure made up of scaled units on which he built his compositions. As he writes, these fractal techniques act as a preparation for composing adding that 'Having made such preparation, then I have found it possible to compose with a kind of intuitive freedom which still assures macrostructural coherence'.

Computer generated fractal music

There have been many instances of computer generated compositions based on fractals as well as computer programmes and software specifically designed to create these. Robert Sherlaw Johnson (1932-2000) was one of the first composers to explore computer generated fractal music. He speculated that 'if meaningful visual patterns can be created by fractal generation, then it should also be possible to create aural ones'. In 'Composing with fractals' he describes the process whereby he takes various iterative formulas to generate musical patterns leading to completed compositions. Echoing the words of Harlan Brothers, he points out that one of the difficulties in composing in this way is that although in visual patterns it is the 'accumulation of values that gives rise to a sensible pattern' in the case of music, 'the whole is not perceived simultaneously and only localised patterns can make sense'. He ends by posing the question 'Can the computer be said to have composed anything?' concluding that in his compositional process although the

244 Michael Frame and Amelia Urry. *Fractal Works. Grown, Built and Imagined.* (Yale University Press, 2016): 104.

computer, using various formulas, generates a stream of variables, 'it is only when the stream of variables is harnessed in a particular way that musical … sense is derived from it otherwise it remains a chaotic sequence'.[245]

So, can fractals be found in music? To recap, in simple terms, a fractal is a geometric figure featuring self-similarity where an identical motif repeats itself on an ever-diminishing scale. So it could be argued that a great deal of music has a fractal element to it in terms of repeated motifs and rhythms used at different pitches or durations, augmentation (scaling up) and diminution (scaling down) being long-established compositional techniques. As we have seen, Wallin criticised what he refers to as Ligeti's 'supposedly 'fractal' music' on the grounds that the techniques that Ligeti uses can be found in traditional compositional practice.[246] Fractals are rarely used systematically, mathematically they are often incomplete or their appearance is only fleeting. Much has been written by musicians about chaos in music, but when aligning compositions with chaos theory, confusion often arises between the everyday definition of chaos as a state of utter disorder or a total lack of organization rather than the scientific definition whereby dynamical systems display unpredictable behaviour at the same time as obeying deterministic laws.

Nested sequences, for example, are necessarily fragmentary because they cannot be sustained across a whole piece. The algorithms used to produce nested sequences could be described as musically mechanical and prescriptive hence they need to be adapted with some creative licence. Of the examples given in this chapter, the prolation canon, the oldest form, seems to be the most consistent with fractal characteristics in its use of motivic scaling and self-similarity. There is a certain irony in this of course: prolation canons were at their most popular over 500 years ago, long before the concept of fractals had been discovered.

---

245 Robert Sherlaw Johnson 'Composing with fractals' in *Music and Mathematics. From Pythagoras to Fractals*. (Oxford: Oxford University Press, 2003).

246 Rolf Wallin. Lecture given at the Nordic Symposium for computer Assisted Composition, Stockholm, 1989.

10.5920/mathsandmusic.12