15 Change ringing

Change ringing is the practice of ringing church bells in a methodical order prescribed by mathematical permutations. This chapter describes some of the mechanics behind bell ringing explaining how these put constraints on the playing of conventional melodies which use changing rhythms and repeated notes. These constraints led instead to the complicated mathematical manner of change ringing. This chapter explores some of the mathematical procedures involved in determining the order in which the bells are rung. During the seventeenth century Fabian Stedman, an English campanologist, wrote the definitive texts on change ringing – *Tintinnalogia* (1668) and *Campanalogia* (1677) - which encompassed several of the mathematical bases of group theory, a discipline that was not properly established until nearly a century later.

Change ringing began in England around the end of the sixteenth century and developed during the seventeenth century in terms of both complexity and popularity. Although the Reformation banned the liturgical ringing of bells, change ringing remained a secular hobby, but not on Sundays. In the second half of the nineteenth century, bell ringing was revived in the church and was again used to call people to worship and to herald important events and church festivals. Change ringing now continues to flourish across the United Kingdom and further afield, particularly in North America and Australia.

Normally there are six to twelve bells in a ring although there may be more or fewer, with 16 being the current maximum.[289] The record for the

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largest number of bells in a church tower is held by Christ Church Cathedral in Dublin which houses 19 bells in its tower. The sequence of bells is usually tuned to a major key. The bells range in weight from around 200 lbs to several tons and are referred to by numbers. The highest bell, known as the treble, is usually numbered one and the lowest, the tenor, is assigned the highest number, with the other bells being numbered in sequence down the scale.

Bell ringers typically stand in a circle each managing one rope. Each of the bells is mounted on a large wheel that swings full circle with the bell sounding when the bell is pointing upwards. Bells can take up to two seconds before they can be sounded again and once a swinging bell is set in motion it is difficult for the bell ringers to vary the interval at which they will ring again. This means that it is difficult to play melodies in the conventional sense because of their repeated notes and changing rhythms. This mechanical constraint led to the creation of change ringing where all the bells are struck in a strict rhythm with an even spacing between one bell sounding and another. The bells are rung in a prescribed order where each bell is rung once moving on to the next prescribed arrangement, essentially a series of mathematical sequences known as changes. Before change ringing was developed a common way to sound a ring of bells was by playing rounds; a repeated sequence of bells descending from the highest to the lowest note. Change ringing now always starts and ends with this sequence.

Rules

To ring the changes means to ring a sequence of changes, whilst obeying three mathematical rules:

- The sequence starts and ends with a round (1 2 3, … n).
- Except for the rounds, as the first and last changes, no change is repeated
- From one change to the next, any bell can move by no more than one position in its order of ringing.

A sequence that includes every possible permutation of bells (or change) is known as an extent. To find the total possible number of changes, the different sequences that can be obtained without repetition, we use the math-

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ematical symbol $n!$ which is referred to as n factorial. It is the product of all positive integers less than or equal to n.

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times 3 \times 2 \times 1$$

So the number of possible permutations on three bells is factorial three

$3! = 3 \times 2 \times 1$, or six changes, six different sequences

123
213
231
321
312
132

and on four bells it is factorial four.

$4! = 4 \times 3 \times 2 \times 1$, or 24 changes;

1234	2314	3124
1243	2341	3142
1423	2431	3412
4123	4231	4312
4213	4321	4132
2413	3421	1432
2143	3241	1342
2134	3214	1324

There are *12!* possible permutations using twelve bells which is calculated thus

$$12! = 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 479{,}001{,}600$$

The value of *n!* grows very quickly.

The table below (Table 1) indicates the numbers of changes in extents on different numbers of bells, and the approximate time needed to ring them. The names are those commonly used by bell ringers for the various sequences.

Table 1

n	**Name**	***n!***	**Time required to ring them**
3	Singles	6	12 seconds
4	Minimus	24	48 seconds
5	Doubles	120	4 minutes
6	Minor	720	24 minutes
7	Triples	5,040	2 hours 48 minutes
8	Major	40,320	22 hours 24 minutes
9	Caters	362,880	8 days 10 hours
10	Royal	3,628,800	84 days
11	Cliques	39,916,800	2 years 194 days
12	Maximus	479,001,600	30 years 138 days

On Wednesday 25th October 2017, twelve bell ringers at St Anne's Church, Alderney on the island of Guernsey, created a new world record when they rang for more than 16 hours – a total of 25,056 changes of Bristol Surprise Maximus. Because of the length of time taken to perform some of these extents, basically any using eight bells or more, it is usual to perform only part of a full extent. The word 'Peal' is used to refer to 5040 changes for seven bells or fewer and for at least 5000 changes for eight bells or more. Typically bell ringers will perform a quarter peal for weddings or other major festivals. This takes 40-45 minutes.

Plain Hunt

The simplest form of generating changing permutations in a continuous way is known as Plain Hunt, a fundamental building-block of many change ringing methods. The word 'hunt' refers to the path each bell makes among the other bells.

Each bell moves one position at each change, unless it reaches the first or last position, where it remains for two changes to enable a turn around. Table 2 below gives examples of Plain Hunt on three, four and six bells. Each row

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shows the order of striking after each change. In each of the sets of changes a line can be drawn along the path of any of the numbers resulting in a straight path from front to back and then from back to front, or vice versa. As can be seen from the table, Plain Hunt produces twice as many changes as there are bells.

Table 2 - Plain Hunt

Three bells	Four bells	Six bells
123	1234	123456
213	2143	214365
231	2413	241635
321	4231	426153
312	4321	462513
132	3412	645231
123	3142	654321
	1324	563412
	1234	536142
		351624
		315264
		132546
		123456

As can be seen in Table 2, with three bells, Plain Hunt produces all six possible changes (factorial three), but with four bells only eight of the 24 possible sequences of factorial four can be produced. This is because of the rule that from one change to the next, any bell can move by no more than one position in its order of ringing. An essential tradition of change ringing is to ring an extent, as many different changes as are possible without repetition, which means that Plain Hunt must be varied to achieve more of the possible changes.

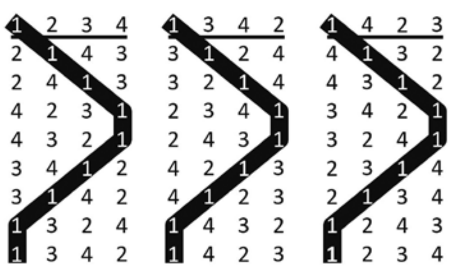
One of the ways that this can be done is through Plain Bob which is based on the Plain Hunt but varied to produce more changes. On four bells Plain Hunt starts from rounds and returns to rounds in eight changes – this length from 1234 back to 1234 is called a lead (see Table 2). In Plain Bob Minimus for four bells, in order to prevent repetition at the end of the first lead the bells in third and fourth place (2 and 4) change places, so instead of ending with 1 2 3 4 the lead ends with 1 3 4 2 – this is called a 'dodge'. The second lead then opens with (1 3 4 2) (see Table 3).

Table 3 – Plain Bob Minimus

First lead	Second lead	Third lead
1234	1342	1423
2143 2413 4231 4321 3412 3142 1324 1342	3124 3214 2341 2431 4213 4123 1432 1423	4132 4312 3421 3241 2314 2134 1243 1234

A second lead is then produced by plain hunting until it reaches its end. Another dodge is made –
1 4 3 2 is changed to 1 4 2 3 which is used at the head of a third lead. The same dodge is used at the lead end to produce a round. In this way, all of the 24 possible changes on four bells – Plain Bob Minimus - have been made without repetition or omission. Table 4 demonstrates how a line can be drawn along the path of the numbers, a straight path from front to back and then from back to front, or vice versa (see below).

Table 4 Plain Hunting in Plain Bob Minimus



It is also possible to ring Plain Bob Doubles on five bells, and Plain Bob Minor on six bells and so on, each of them using dodges and having the same basic structure as Plain Bob Minimus.

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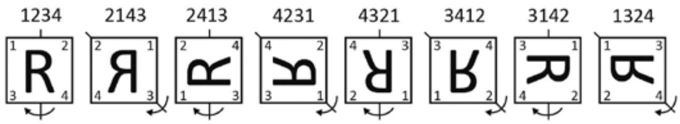
During the seventeenth century Fabian Stedman (1640 – 1713) an English author and campanologist, wrote the definitive texts on change ringing, the first two publications on the subject, where he outlined the methods and rules on creating changes along with instructions on how to follow them. His book *Tintinnalogia – or the art of change ringing* (1668) was followed by *Campanalogia* (1677). Through his theoretical ideas and compositions he was implicitly drawing on several of the principles of group theory, a mathematical discipline that was not properly established until nearly a century later. These ideas akin to group theory include symmetry groups, permutation groups, cosets, and factorials and appeared many decades before the definition of an abstract group as a set with a binary operation satisfying certain axioms was established.[290 291]

The following diagram illustrates how the eight permutations in the first column of Plain Bob Minimus correspond to the eight symmetries of a square. It also illustrates how the transitions between the permutations correspond to flips of the square (see below).

Plain Bob Minimus. The eight symmetries of the square



As a collection, the symmetries of the square form a mathematical structure known as the symmetry group of the square. This means that the separation of Plain Bob Minimus into three leads corresponds to the partition of the general symmetric group on four elements into the cosets (subgroups) of that group.[292]

290 A group is defined as a set with a binary operation if the following axioms are satisfied: closure, associativity, , identity and inverse.

291 To read more about Stedman's links with group theory see Arthur T. White. 'Fabian Stedman: The First Group Theorist?' *The American Mathematical Monthly*, Vol. 103, No. 9 (Nov., 1996): 771-778.

292 https://plus.maths.org/issue53/features/polsteross/2pdf/index.html/op.pdf Burkard Polster and Marty Ross. 'Ringing the changes'.

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Change ringing is rooted in several different mathematical concepts. The number of possible permutations for a defined number of bells is calculated by using *n* factorials. The sequence of permutations, the changes, are generated by systematically following a set of rules which could be described as algorithmic in that precisely described instructions are designed to be applied systematically through to a conclusion in a number of steps. Furthermore, the art of change ringing draws on several of the principles of group theory.

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