8 Serialism and the 12-note scale

PERHAPS MORE THAN any other technique of composing, serialism is rooted in mathematical concepts. It is algorithmic in its use of a strict set of procedures to generate a note row and it uses symmetrical transformations in the construction of different rows. This chapter looks at the procedures of serialism in the music of its originator, the Austrian composer Arnold Schoenberg (1874 – 1951) focussing on his *Variations for Orchestra*. It goes on to examine the different compositional approaches of the other two members of what is known as the Second Viennese School[117] - Alban Berg (1885 – 1935) and Anton Webern (1883 – 1945). The next sections look at extensions of serial approaches: rhythmic serialism where patterns of durations are ordered, and total serialism where algorithms are applied to further elements of the music such as dynamics and attack. The chapter ends with a more detailed exploration of both mathematical and musical considerations with a discussion of whether the two can be reconciled.

The twelve-note system of serialism (sometimes known as dodecaphony) follows strict mathematical rules in the form of an algorithm. First of all a note row (or series) is composed - this uses all of the 12 notes of the chromatic scale and is the basis of the whole composition. None of the notes are repeated in the note row (also known as the prime order or prime series). The note row can exist at each of the transpositional levels, as can each of the inversion, retrograde and retrograde inversions. This means that the row

117 The Second Viennese School is an umbrella term for the composers Schoenberg, Berg and Webern. It is sometimes understood to include other pupils of Schoenberg. It is based on the tenuous notion that there was a First Viennese School which might have included Haydn, Mozart, Beethoven and Schubert.

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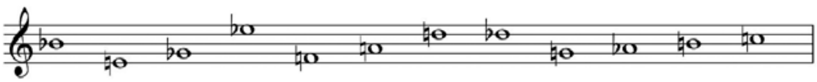
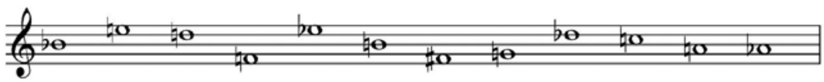
can be used in any of its 48 forms. It can also be transformed through scale symmetry in two ways: augmentation and diminution. The durations of the original notes can be lengthened (augmentation), or shortened (diminution). The original notes can also be used in another octave (octave displacement). There are four main permutations of the row, each of these is rooted in symmetrical transformation. They are often referred to as the prime order, inversion, retrograde and retrograde inversion.

Here are the four permutations of the note row used in the first movement (Introduction and Theme) of Schoenberg's *Variations for Orchestra, Op. 31* (1928), his first twelve-tone composition for a large ensemble.

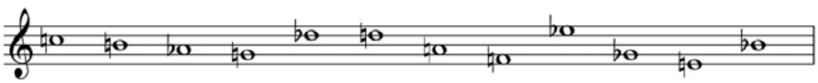
Prime order – P



Inversion – I uses horizontal reflection, each of the intervals is turned upside down



Retrograde - R uses vertical reflection where the note row is played backwards



Retrograde inversion – RI. The inversion can be played backwards. In other words it undergoes a 180^0 rotation.

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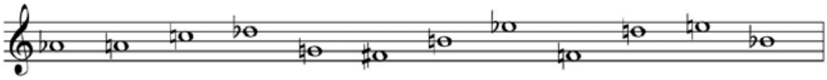
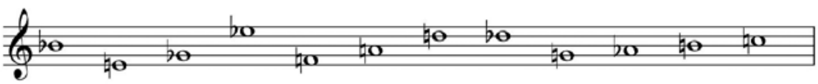


All the material of the *Variations for Orchestra* is derived from the four versions of the row. Following the 'Introduction', the second section of the *Variations for Orchestra*, 'Theme' (bars 34-57) presents four linear statements of the theme in the order P, RI, R, I. Schoenberg marks this with the word '*Haupstimme*', a term he used to denote the principal part in a complex texture.

In addition to the symmetrical aspects, the notes of the row can also be used vertically to make chords (sometimes known as verticalisation). Schoenberg excluded the use of consonant chords because of their tonal qualities. It can be seen from the examples below that the Prime row and the Retrograde Inversion (in transposition) are both made up from the same two hexachords – Bb E Gb Eb F A and D Db G Ab B C.

Prime order – P



Retrograde inversion – RI in transposition

Together these two hexachords form a 12-note aggregate. The series is therefore said to show the property of 'combinatoriality', a technique found in twelve-note compositions whereby a collection of pitch classes (in this case P) can be combined with a transformation of itself (in this case RI) to form an aggregate of all 12 pitch classes. The term 'combinatorics' has been borrowed from mathematics where it refers to the study of the enumeration,

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combination and permutation of sets of elements and the mathematical characteristics of their properties.

Alban Berg

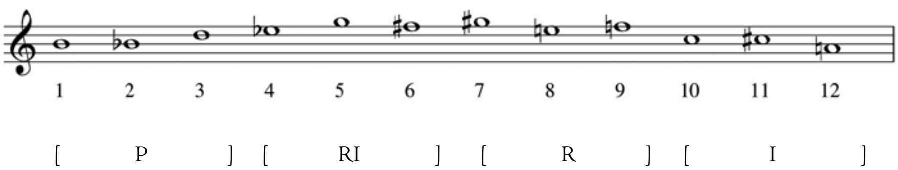
Each of the three members of the Second Viennese School took a different approach to serialism. Alban Berg, a pupil of Schoenberg, was quite free in his use of the note row reconciling twelve tone procedures and tonal harmony. His music tends not to keep to a single series, even within a movement. He also incorporated tonal elements into the twelve-tone language, notably in his Violin Concerto (1935) where the original row uses eight ascending thirds followed by a note row and consequently spells out two major and two minor chords

1 + 2 + 3 = G minor, 3 + 4 + 5 = D major, 5 + 6 + 7 = A minor, and 7 + 8 + 9 = E major.
8 – 12 = five notes of a whole tone scale.

The Violin Concerto also uses a Bach chorale taken from his Cantata No. 60 and a Corinthian folk tune, both tonal in nature.

Anton Webern

Webern, who also studied with Schoenberg, took a much stricter approach to the dodecaphonic rules: unlike Berg, he always used a single series for each composition and suppressed all repetition of material. He constantly exploited the possibilities of the note row with much use of concentrated motivic working and symmetrical structures. In his *Concerto Op. 24* (1934) this symmetry can be seen within the row itself where the notes 1 - 6 and similarly 7 – 12 can be reflected along the middle to give a mirror image where the same order of intervals appears forwards as well as backwards (see below).



The order of intervals between the first six notes is as follows

B to Bb (semitone) Bb to D (major 3^{rd}) D to Eb (semitone) Eb to G (major 3^{rd}) G to F# (semitone)

In reverse order (retrograde) the notes F# G Eb D Bb B follow the same pattern - semitone, major 3^{rd}, semitone, major 3^{rd} semitone.

Furthermore if the notes 1- 3 are regarded as a microcosmic Prime version, then 4- 6 is the Retrograde Inversion, 7 – 9 is the Retrograde and 10 – 12 is the Inversion.

Here are the opening bars of Webern's *Concerto Op. 24* which illustrate how Webern has used the row in harmony in his composition. The parts enter in this order – oboe (P), flute (RI), trumpet (R) and then clarinet (I). The piano enters in bar 4 where the overlapping parts follow this order – P (retrograde), RI (retrograde), R (retrograde), I (retrograde). Notice how various forms of the row are unfolded simultaneously, beginning on different beats of the bar and interlocking (see below). This all helps to create a complex rhythmic structure, a characteristic of Webern's compositional style.

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In his *Symphony Op. 21* (1928), and all the works that followed, the twelve-tone technique was used with unprecedented strictness and the concept was extended to include, not just pitch, but also timbre and rhythm. In Webern's words,

> So what has in fact been achieved by this [twelve-note] method of composition?… What territory, what doors have been opened with this secret key? To be very general, it's a matter of creating a means to express the greatest possible unity in music.[118]

Serialism with other pitch-class collections

Before he arrived at his twelve-tone system, Schoenberg compositions included series containing a number of pitch classes greater or lesser than

118 Letter from Anton Webern to Alban Berg cited in *Die Reihe 2: Anton Webern*. (London: Universal Edition, 1960): 42.

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12. In the *Five Piano Pieces Op. 23*, for example, No. 2, uses a series of nine different pitch classes and No. 3 is based on a 14-note series. Similarly, before Stravinsky fully adopted 12-note serialism in *Threni* (1957–8) he used series with fewer than 12 pitch classes; *In memoriam Dylan Thomas* (1954), for example, uses a series of only five pitch classes. There are also examples of series with more than 12 pitch classes which include Berio's *Nones for orchestra* (1954) which is based on a 13-note series made up of two overlapping heptachords, the second being the retrograde inversion of the first.[119]

Rhythmic serialism

As Paul Griffiths points out 'interpretations of rhythmic serialism are problematic. There is no equivalence class to correspond with the pitch class … there is no analogue for inversion'.[120] An early example of the use of ordered patterns of durations can be found in the third movement of Berg's *Lyric Suite* (1926) where he creates a rhythmic series made up of 12 durations, each of one, two or three units. The series is used in exact retrogrades, and is varied by adding rests and 'transposed' by multiplying all the values by the same integer. In 1940 Webern composed his *Variations Op. 30* for orchestra which uses versions of two rhythmic motifs presented in the opening bars and derived from the note row. These are in effect sets of durations.

The American composer Milton Babbitt (1916-2011), a trained mathematician, demonstrated and formalized many aspects of 12-tone compositional technique in several important essays where he introduced algebra and number theory to model 12-note rows and operations. In 'Some Aspects of Twelve-Tone Composition' (1955), 'Twelve-Tone Invariants as Compositional Determinants' (1960), and 'Set Structure as a Compositional Determinant' (1961) he introduced terms derived from mathematics such as 'source set', 'aggregate,' 'secondary set', and 'derived set.' These terms were used in the classification of different types of pitch class set and helped to describe the different procedures used by such sets. He coined the term 'combinatorial' to describe 12-tone sets that yield aggregate and secondary

119 Paul Griffiths in 'Serialism' in *Grove Music*

120 Ibid.

set formations. This nomenclature has since become widely adopted as the basis for theoretical work notably in music set theory.[121]

In 'Twelve-Tone Rhythmic Structure and the Electronic Medium' (1962) Babbitt demonstrated a variety of methods for interpreting the structures of pitch class sets in the rhythmic domain.[122] This included an analogy between the octave (in pitch structure) and the bar (in rhythmic and metrical structure) by dividing the bar into 12 equal units each of which can be articulated by individual points of attack. In this way pitch class sets can be mapped onto 'time-point sets' where a set of integers can be interpreted both as a pitch class set and as a time-point set, defined as the duration which separates the attack from the beginning of the bar. So, for example, 0, 11, 6, 7, 5, 1, 10, 2, 9, 3, 4, 8 can be interpreted as a pitch class set where the numbers are generated by counting up in semitones from the first note (designated as 0) (see below).[123]

or as a time-point set a particular point of attack is a measure of its position within the bar. In the example below the metrical unit is a demi-semiquaver, a 12th of the whole bar. Here the 12 available points of attack within a bar are ordered according to the numerical set given above. Time-point 0 occurs on the first demi-semiquaver of the bar, time-point 1 on the next, and so on (see below).

121 Set theory in music originated with Milton Babbitt and was then formulated in 1973 by the American music theorist Allan Forte. This led to more rigorous theoretical formulations by John Rahn, Robert Morris and David Lewin who applied mathematical group theory to music.
122 "Twelve-Tone Rhythmic Structure and the Electronic Medium," *PNM*, i/1 (1962), 49–79; repr. in *Perspectives on Contemporary Music Theory*, ed. B. Boretz and E.T. Cone (New York, 1972), 148–79.
123 Elaine Barkin, revised by Martin Brody and Judith Crispin. 'Milton Babbitt' in *Grove Music*

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Babbitt used this method in his *Second String Quartet* (1954). The time-point series is manipulated to produce different versions of the rows: inversion by complementation of the time-point numbers to 12; transposition by the addition of a constant value to each time-point number; and retrograde by using the time-point set backwards.[124]

Total serialism

In the early 1950s, several composers including Boulez, Stockhausen, Maderna and Nono extended the procedures of serialism to the other aspects of the music beyond pitch, such as rhythm, dynamics, tempo, timbre and note attack. This method has become known as total serialism or integral serialism. These composers were much associated with the Darmstadt School, a loose grouping of composers associated with the International Summer Courses for New Music in Darmstadt, West Germany.

An early example of total serialism can be found in Babbitt's *Three Compositions for Piano* (1947). Here points of articulation are determined by a set along with dynamics and pitch set classes. In 1949, Messiaen composed his piano piece *Mode de valeurs et d'intensités* (Mode of Durations and Intensities) which was based on scales, not only of pitch but also of duration, attack and dynamics. Two years later Boulez based his piece *Structures 1a* (1951) entirely on the note row used by Messiaen in *Mode de valeurs et d'intensités*. From this he created two number matrices which represented all 48 versions of the row. An integer was assigned to each pitch class, so for example Eb = 1, D = 2 and so on. From these matrices he derived another system to determine the durations of notes where the integers were read as numbers of demi-semiquavers (sixteenth notes). In addition, each statement of the row was given a particular dynamic

124 Griffiths, 'Serialism', *Grove Music.*

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and mode of attack.[125] These matrices could be described as a pair of magic squares (see Chapter 14 for more detail).

Another leading figure in the serialist avant garde was Karlheinz Stockhausen (1928-2007), himself a teacher at Darmstadt. In his works from 1953 onwards, serial organization permeates every level of the formal process. His works *Kreuzspiel* (1959) and the *Schlagtrio*, are examples of 'total serialism' where the basic series is permutated and applied to durations, dynamics and articulation as well as pitch. In his *Elektronische Studien* (1954) he applied similar rules to the organization of pitch, rhythm and timbre. His work *Gruppen* (1957) for three orchestras (combining 109 players) and three conductors explores the possibilities of different simultaneous tempos using sets of tempi.

Some mathematical considerations

Serialism draws heavily on mathematical principles as we have seen above: it is algorithmic in its use of a strict set of procedures designed to be applied systematically; it uses symmetrical transformations in the construction of different rows (vertical and horizontal reflection, translation and 180^0 rotation) and it uses combinatorics in its formulation of harmonies.

Babbitt's theoretical writings about 12-note music eventually led to the formulation and development of music set theory which has some parallels in mathematical set theory and group theory. In music a set is a group of pitch classes, usually a 12-note set, it may also be a set of other musical elements such as durations or dynamics. Music set theory assigns integers to different pitch classes, categorises musical objects and analyses their relationships through operations such as transpositions and inversions.

Because the series can use transpositions of the 12 notes of the chromatic scale, as can each of the inversion, retrograde and retrograde inversions, the original series can exist in 48 different forms. However it is not always the case that a note row will generate 48 different forms. There are some rows which will generate the same row under certain operations, particularly if

125 Jonathan Cross. 'Composing with numbers' in John Fauvel, Raymond Flood and Robin Wilson. *Music and Mathematics.* (Oxford: Oxford University Press, 2003): 137 and Paul Griffiths in 'Serialism' in *Grove Music*.

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the original row demonstrates some kind of symmetry. For example, if we were to take a descending chromatic scale beginning on C as the note row then the Prime order would be as follows.

The Inversion would be

and the Retrograde Inversion would be

which is, of course, the same as the original row. In other words it is invariant. In mathematics, an invariant is a property of a mathematical object which remains unchanged after operations or transformations are applied to the objects.

To find the total possible number of note rows, we use the mathematical symbol $n!$ which is referred to as n factorial. It is the product of all positive integers less than or equal to n.

$n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times 3 \times 2 \times 1$

There are 12! possible permutations of the row which is calculated thus

$12! = 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 479{,}001{,}600$

Another, more tenuous, connection to mathematics has been made to Einstein's 1915 Theory of General Relativity. In his book *Music by the Numbers*, Eli Maor argues that a comparison can be made between Schoenberg's serialism and Einstein's theory of relativity where all systems of reference are equal to each other, the democracy of the twelve notes and their lack of hierarchical relationships meant that 'Henceforth only the position of each note relative to its immediate predecessor would matter; you might call it relativistic music'. Maor quotes the words of the composer Pierre Boulez 'With it [the twelve-tone system], music moved out of the world of Newton and into the world of Einstein'[126] Maor's argument, however, is lacking in detail. Although it is true to say that at the beginning of the twentieth century, Schoenberg's new system created a profound revolution in music just as Einstein's Theory of Relativity shook the foundations of science, there is little or no evidence to suggest that Schoenberg's twelve-tone system was influenced by the work of Einstein. When speaking about any potential link, Schoenberg pronounced that 'There may be a relationship in the two fields of endeavour but I have no idea what it is. I write music as music without any reference other than to express anything but music'.[127] Einstein in turn was baffled by Schoenberg's music which he described as 'crazy'.[128]

Some musical considerations

When Schoenberg invented his twelve-note system he abandoned the tonal system which had been used in music for the previous 300 years. In comparison with the hierarchies of the tonal system, serialism took a democratic approach where all 12 notes of the chromatic scale were equal and no note was bound to the home key. Tonality emphasised the importance of the tonic and dominant and every note had a specific relationship to the tonic. Put simply, in the words of the composer Christopher Fox, 'The overriding characteristic of the series is that it is non-hierarchical – everything is as

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126 Eli Maor. *Music by the Numbers. From Pythagoras to Schoenberg*. (Princeton: Princeton University Press, 2018): 125.

127 https://www.schoenberg.at/index.php/de/1933-r-p-arnold-schoenberg-receives-the-american-press

128 Andrew May. *The Science of Sci-Fi music*. (London: Springer, 2020): 88.

important as everything else …'[129] The algorithmic nature of serialism with its set of mathematical rules led some to see the method as the compositional equivalent of painting by numbers. As Jonathan Cross puts it 'Accusations of lack of artistry, lack of creative imagination, and even lack of musicality have been hurled by critics and music-lovers alike at very many twentieth-century composers not least at the Viennese composer, Arnold Schoenberg'.[130]

The new system of serialism raised several questions, not least whether it could be perceived within the music by the listener or whether instead it was incoherent with a limited capacity to communicate the composer's thoughts. The musicologist Fred Lehrdahl maintains that 'if the serial construction is so important – and yet it is not an explicit part of the listeners' experience, the listeners' experience must be impoverished'.[131] In his book on aesthetics, *Music, the Arts, and Ideas*, the philosopher Leonard B Meyer argues that listeners are conditioned to the perception of tonal music making it more difficult to perceive serial pieces, a situation compounded by the fact that composers often used quite different serial procedures in different pieces of music, and that a satisfactory perception of serialism in one work would not necessarily help the listener in others.[132] In tonal music the hierarchical relationships between pitches in the diatonic scale help to establish and define the listener's expectations. The counter arguments from serial composers usually adopt the position that serial procedures are not intended to be perceived by the listener or that although they are not perceived consciously, the music nevertheless gives an effect of coherence which the listener cannot explain.[133]

Many works using serial techniques were premiered at the Darmstadt International Summer Courses for New Music and were subject to much debate.[17] Such discussions tended to foreground the technical analysis of the music rather than the expressive purposes to which it was applied. Perhaps the final word should go to Christopher Fox who, in his essay 'Darmstadt

129 Christopher Fox. 'Music after Zero Hour' *Contemporary Music Review*, Vol. 26, No. 1, 2007: 16.

130 Cross, 'Composing with numbers', 131.

131 Fred Lehrdahl 'Cognitive constraints on compositional systems'. *Contemporary Music Review*, Vol. 26, No. 1, 1992: 97-121, cited in Christopher Fox. 'Darmstadt and the Institutionalisation of Modernism' *Contemporary Music Review*, Vol. 26, No. 1, 2007: 120.

132 L.B. Meyer: *Music, the Arts, and Ideas* (Chicago and London, 1967)

133 Griffiths, 'Serialism', *Grove Music*.

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and the Institutionalisation of Modernism', argues that 'generally the aesthetic purpose of Darmstadt serialism is not that the series should occupy the musical foreground; rather, it is that the serial principle is the method whereby musical transformations can be achieved.'[134]

134 Christopher Fox, 'Darmstadt and the Institutionalisation of Modernism', *Contemporary Music Review*, Vol. 26, 2007: 120.

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