9 Canons and fugues

THIS CHAPTER FOCUSES on two musical forms, the canon and the fugue, where geometric transformations and imitation are central to their composition. The word canon is used to refer to both a musical form and a compositional technique where an initial melody is imitated at a specified time interval by one or more parts. A section of a piece is described as canonic if the imitation prevails throughout the section, if the imitation is exact and continues through the whole piece, it is called a canon. As a strict form canon makes use of different types of geometric transformation: reflection, translation, scale and, less frequently, rotation. Canons fall into several discrete categories which are outlined in this chapter. These are distinguished in different ways: by the number of voices used; imitation at different distances; imitation at different upper or lower intervals; and by the geometric transformations used. The fugue, a more complicated form, follows a detailed set of rules; the composition of fugues has often been regarded as one of the most highly-skilled compositional techniques.

One of the earliest examples of the canon as a musical form is 'Sumer is Icumen In' which dates from the mid-thirteenth century. Four voices sing the same melody one after the other, accompanied by two lower voices (the *pedes* meaning 'foot') which themselves alternate and overlap. A culmination of 14th-century canonic technique was reached in the works of Guillaime de Machaut (c1300-1377), his rondeau 'Ma fin est mon commencement' is the earliest known piece based on retrograde procedures (see pages 63-4). The canon had its heyday during the late fifteenth and early sixteenth centuries with the works of composers such as Guillaume Dufay, Johannes Ockeghem and Josquin des Prez. During this period, mastery of canonic technique was equated with proficiency in composition; practical treatises

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of the 16th century often included compendia of canonic devices. Johannes Ockeghem (c1410-1497) almost certainly invented the prolation canon where each melodic line uses the same music but played at different speeds and is one of the most difficult types of canon to execute.[135]

Canon as a learned form of counterpoint lost much of its popularity during the Enlightenment: it was seen more as an outdated academic exercise which placed ingenuity ahead of musical value and freedom of expression. J S Bach (1685-1750), however, used canon throughout his career notably in the *Musical Offering*, the *Goldberg Variations*, the canonic variations of *Vom Himmel hoch* and the *Art of Fugue*. In these works the form hit a new peak and, chiefly through Bach's influence, the canon once again became an important musical device.

The use of canonic principles, although less popular, carried on and examples can be found in the works of Haydn, Mozart, Beethoven, Brahms and Mahler.[136] The twentieth century saw a renewed interest in canonic principles: its use of geometric transformations was central to both serialism and minimalism. There are many examples of canons and canonic writing in the music of Schoenberg, Stravinsky and Anton Webern.

Rounds and catches

A straightforward form of canon is the 'catch' or 'round' where each of the voices enters one after the other with the same melody creating a polyphonic texture.[137] Because all the voices sing the same tune, rounds are relatively easy to sing; well-known examples are the children's songs 'Row, row, row your boat' and 'Frère Jacques'. A single-line melody is constructed so that a round forms its own harmony when sung at the unison, consequently they are frequently constructed around only one or two chords to make the compositional process easier. During the seventeenth century, rounds,

135 John McDonough and Andrzej Herczynski. *Chaos, Solitons and Fractals, Science Direct 170 (2023).*

136 A well-known example of a canon can be found in the third movement of Gustav Mahler's *Symphony No. 1* (1888) which is based on a minor version of 'Frère Jacques'

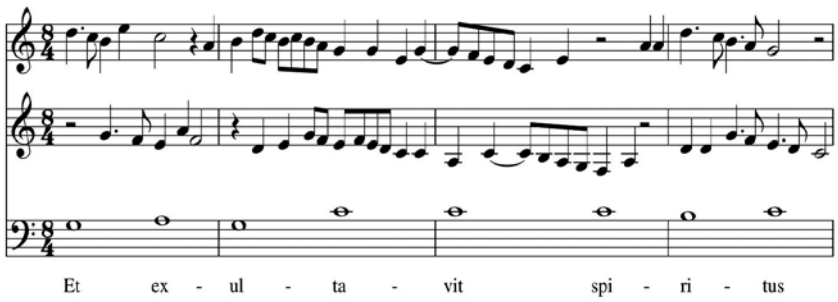
137 In some ways the terms 'round' and 'catch' are interchangeable but 'catch' is often used to refer to comic, sometimes bawdy, English rounds written between roughly 1580 and 1800.

often drinking songs with bawdy words, were largely regarded as light entertainment being sung in clubs and taverns – usually by men, frequently on the topics of drinking and sex, and often with lewd words. Henry Purcell (1659-1695) contributed many such bawdy songs such as 'Once, twice thrice'. In this context, they were usually referred to as catches.

The main types of canon

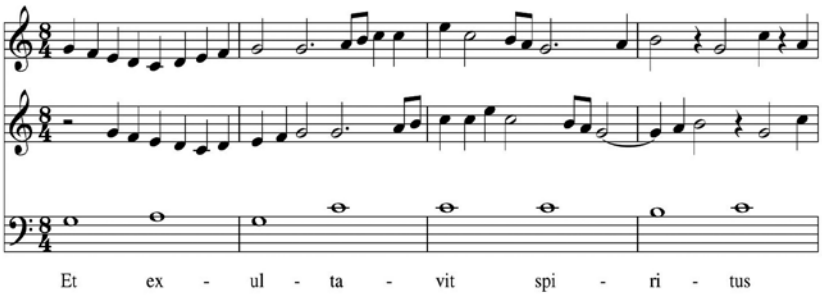
Canon in unison is one in which the imitating part(s) enters at the same pitch as the leading part. This is the most common form and is often found in folk music around the world, a well-known example being 'Shalom Chaverim'. Canon at the octave is one in which the imitating part(s) enter an octave apart from the leading part. Canon at the fifth is one in which the imitating part(s) enter a fifth apart from the leading part. Entries at any other interval are similarly explained.

The following two examples are taken from the first volume of *Tabulatura nova* (1624) by Samuel Scheidt (1587-1684). In both, the two canonic parts form contrapuntal lines upon a *cantus firmus* taken from the Magnificat.[138] The first is an example of canon in unison where the imitating part follows the leader a quarter of a bar later at the same pitch (see below).



138 *Cantus firmus* is a Latin term referring to a pre-existing melody used as the basis of a new polyphonic composition. It is found in Medieval and Renaissance music.

The second example is of canon at the fifth where the imitating part follows the leader a quarter of a bar later a fifth lower (see below).



The thirty harpsichord variations of the *Goldberg Variations* by J S Bach (1685-1750) are grouped in threes where each trio of variations culminates in a canon. The canons themselves progress sequentially according to intervals. The first canon is at the unison, the second canon at the second, the third canon at the third and so on through to variations 25–27, which culminate in a canon at the ninth.

In **canon by inversion** (sometimes referred to as *Al rovescio*) the imitating part plays the melody in inversion: an upward interval becomes a downward one and vice versa. This use of inversion can be seen in the opening bars of Canon No. 2 in *Fünf Canons, Op. 16* (1924) by Anton Webern (1883-1945). The five pieces are composed as strict canons and the texts themselves could be regarded as 'canonical', given that they are taken from Latin liturgical texts. The work is scored for soprano, clarinet and bass clarinet. Canon No. 2, *Dormi Jesu*, is for soprano and clarinet. Notice how the inverted intervals are precise. The clarinet opens with a leap of an augmented 5th (nine semitones) from Bb up to F# whereas the soprano opens a bar later with a descending minor 6th (nine semitones), E down to G#. The next two clarinet notes rise up a minor 3rd, A to C, whereas the singer follows with a falling minor 3rd F down to D.

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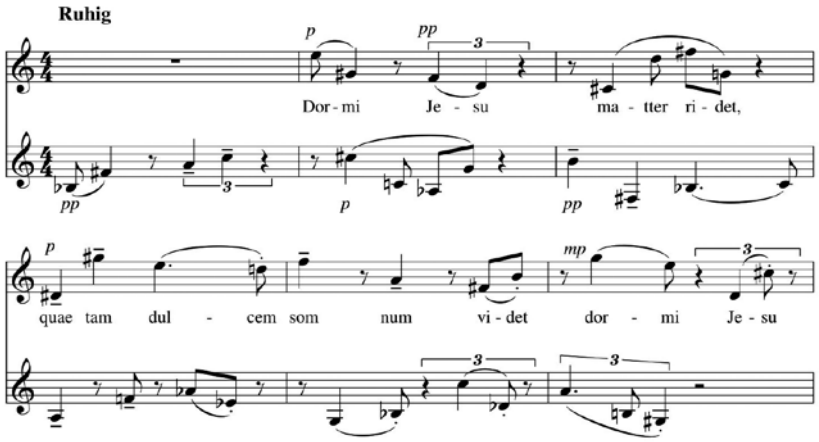
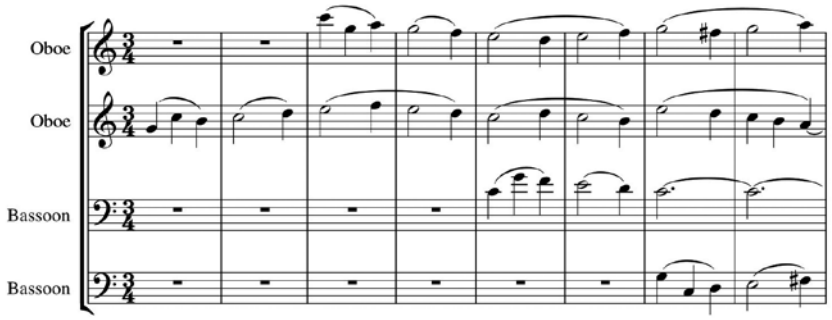
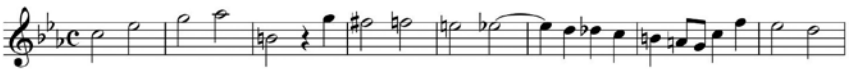


Double canon uses two melodies which are given out at the same time by two parts and are imitated by two other parts: in other words, two two-part canons proceed simultaneously. The Trio of Wolfgang Amadeus Mozart's *Serenade No. 12 in C minor* (1728) is a double canon using two melodies in inversion for two oboes and two bassoons (see below). It is marked 'Al rovescio'; Oboe II leads with Oboe I entering two bars later in inversion. Bassoon 1 leads the second canon with Bassoon II entering two bars later also in inversion.



Symphony Op. 21 (1928) by Anton Webern makes much use of mirrors and palindromes through a series of double canons.

Retrograde canon is a type in which the imitating part presents the melody backwards. It is also known as canon cancrizans or crab canon (even though crabs move sideways rather than backwards). Unlike most other canons, the entries are not staggered. Instead both parts start at the same time. Crab canons are relatively uncommon. One of the most famous crab canons is included in J S Bach's *Musical Offering* (1747). This is a collection of pieces based on what is known as the Royal Theme, so called because it was presented to Bach by Frederick the Great, himself a composer and flautist, as a subject for an on-the-spot improvisation. 139 Here is the Royal Theme.



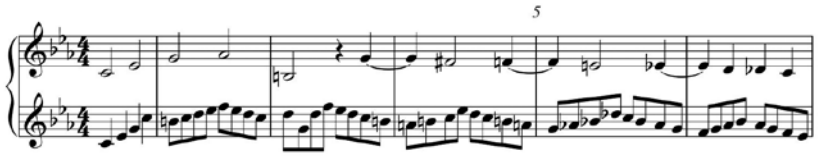
The *Musical Offering* includes 10 canons of different types. These were notated as puzzle canons in the original printed edition, and the full parts were not transcribed. A puzzle or riddle canon is one in which the canon is notated in an enigmatic way where the composer gives only the leading voice along with cryptic instructions for its resolution. It is then up to the performer to solve the intellectual exercise in order to discover the correct time and interval distance for the voice(s).[140]

Three of the canons in the *Musical Offering* have no title or instructions as to when the imitations should be played by the follower. Instead they have the Latin inscription *Quaerendo invenietis* (Seek and ye shall find) with clues provided within the music. So, for example, in No. 1, Canon *a 2 cancrizans*, the title tells us that the canon is for two voices and it should be played backwards. Furthermore at the end of the main melodic line, the clef and key signature were originally written facing backwards. The Royal Theme is presented in the first five bars of the upper part (see below). The lower part is an exact retrograde of the upper part and the upper part is an exact retrograde

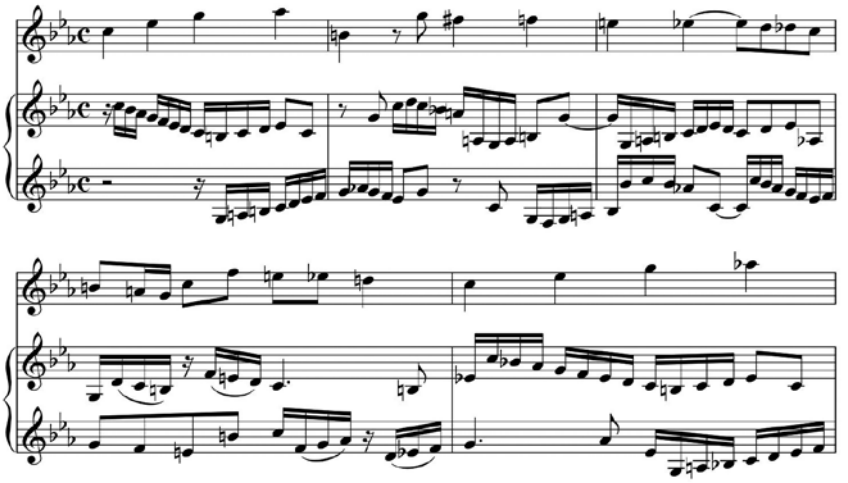
139 Webern used this royal theme in his 1935 orchestral arrangement: *Ricercare Fuga (Ricercata) a 6 voci* from J S Bach: *Musical Offering*).

140 Two of Bach's former students, Johann Philipp Kirnberger and Johann Friedrich Agricola are credited with solving these in the *Musical Offering* as cited in http://www.early-music.com/js-bach-musical-offering/ and Gareth E Roberts. *From Music to Mathematics*.(Baltimore: John Hopkins University Press, 2016): 178.

of the lower part. The point of symmetry, a vertical reflection, is between bars 9 and 10. This canon as well as displaying symmetry has another interesting mathematical aspect: it can be visualised on a Möbius strip. A Möbius strip is a 2-dimensional strip where one continuous side is formed by joining the ends of a rectangle after twisting one end through 180°. The canon should be written as one long line of music on a strip of paper and then cut in half at the end of bar 9. It should then be glued together. It could then be played by two instruments travelling in opposite directions around the strip. The twist is the point at which the two parts interchange.

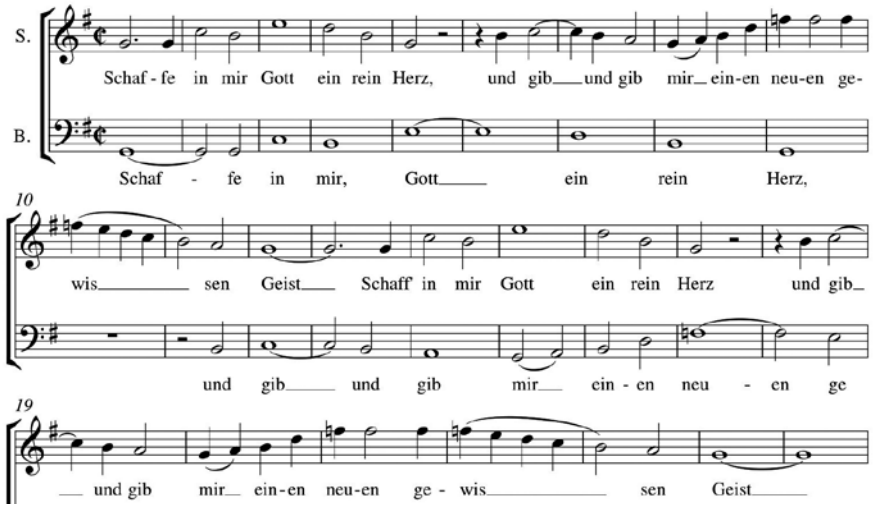


Mirror canon usually refers to a canon in which the combined principles of inversion and retrograde are used in the imitating part. In other words, an inverted entry moves in contrary motion – in the opposite direction. Sometimes it is used to refer to canons in inversion. Canon 3 (a 2 *per Motum contrarium*) in J S Bach's *Musical Offering* is a mirror canon at the fifth. Here the Royal Theme is used as an accompaniment and is not treated canonically. It can be found in the upper voice, but using diminution where the original note values are halved (and consequently played twice as fast). The title tells us that the follower sings the leader's theme in contrary motion. It is also played backwards (see below).

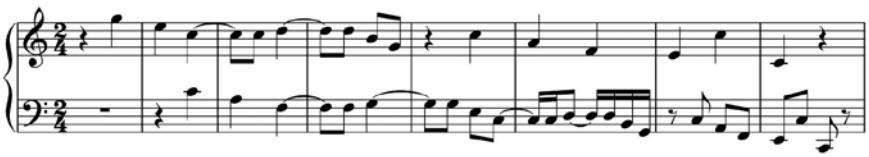


Canon by augmentation has the imitating parts in longer notes than the leading part. The notes are usually twice as long.[141] The following example is the taken from the opening bars of the first movement of the Brahms motet *Schaffe in mir, Gott, ein rein Herz, Op. 29 No. 2*, a setting of Psalm 51. Here the bass II part imitates the soprano part, this time at the same pitch but in augmentation. The note values are twice as long. This means that the bass part takes twice as long, consequently the soprano line is repeated. The example below shows the entire canon at the octave in augmentation between the Soprano and Bass II. To make this clearer the other voices have been omitted.

141 Canon by augmentation is sometimes known as a sloth canon.



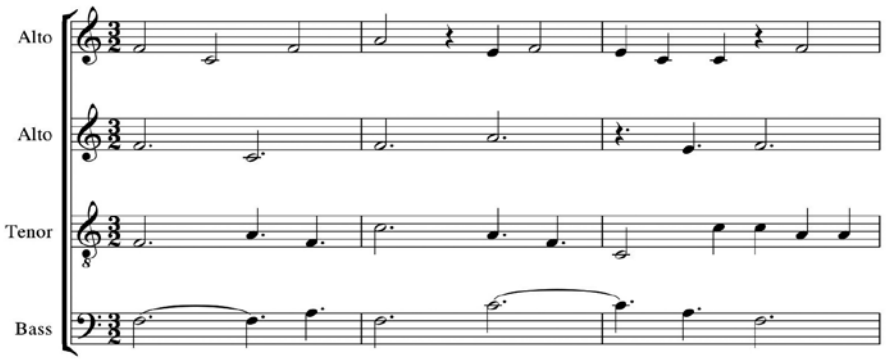
Canon by diminution has the imitating parts in shorter notes than the one that they are imitating. The notes are usually half as long. In his *Miscellanea Musica*, C P E Bach (1714-1788) composed several canons, partly for teaching purposes. These include 'Diminution canon for two voices in C major'. Examples of canon by diminution are rare. This is because the voice with the shorter note values soon overtakes the 'slower' one with its longer note values. C P E Bach avoided this in his 'Diminution canon' by beginning the piece with straightforward imitation at the lower fifth and then from bar 5 imitating the leader (two octaves lower) but with half the note values - effectively at twice the speed.



It should be noted that in the above examples the transformational devices used are often unrecognisable to the listener unless they have analysed the score first. Although the main melodic themes can act as a unifying factor, holding the music together, the transformations are often easier to see than to hear.

Prolation canons

A prolation canon combines two or more identical musical lines at different rates; each voice sings or plays the same music, but at different speeds. Voices may either enter successively or simultaneously. The concept is simple but the compositional technique is difficult, hence prolation canons are quite rare. The earliest master of the prolation canon is Johannes Ockeghem. Each movement of his *Missa Prolationum*, dating from the second half of the fifteenth century, uses a different variant of this form where there are different gaps between the entries and different relative speeds of the voices. 'Kyrie Eleison I' is a double prolation canon, where two separate pairs of voices, soprano/alto and tenor/bass, take up a different motif. The ratio of tempos between the soprano and the alto is 3:2. Each of the first six note values of the soprano part is multiplied by 3/2 in the alto part. After the opening passage shown below, the music continues as a regular canon.

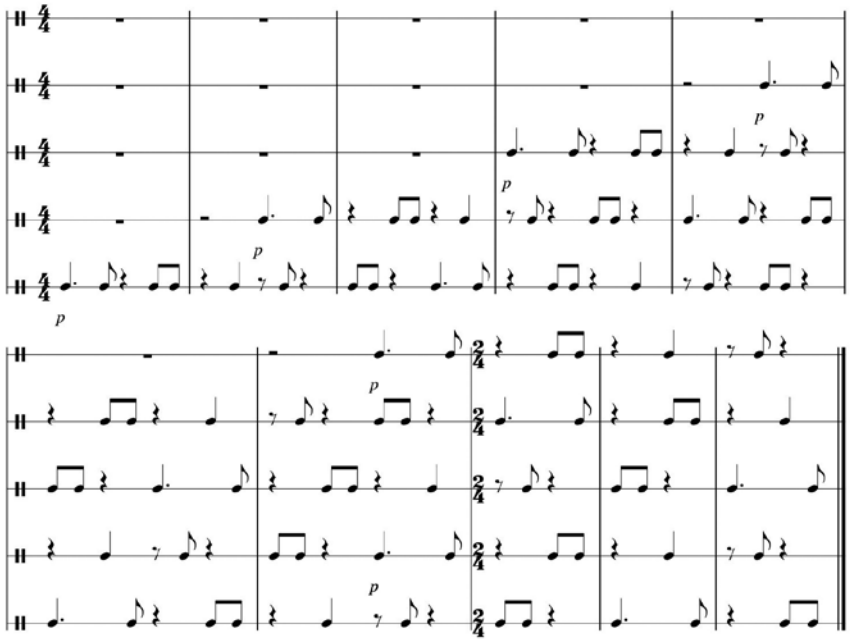


Other examples of prolation canons include J S Bach's *Canon a 4 per Augmentationem et Diminutionem*, the final movement of *14 Canons, BWV 1087,* which is built on a repeating bass line taken from his *Goldberg Variations*. The Estonian composer Arvo Pärt (b 1935) uses a prolation canon in his vocal work *Cantus in Memory of Benjamin Britten* (1977) which was used in the 2004 film *Fahrenheit 9/11*. Here a series of falling A minor scales is heard across five voices with each group entering an octave lower and at half the speed. Another relatively recent example can be heard in the first

movement of Shostakovitch's *Symphony No. 15*. It has been argued that prolation canons are examples of fractals in music (see pages 199-200).

Rhythmic canons

The final type of canon to be explored is the rhythmic canon, one that is relatively rare although used by both Alban Berg and Olivier Messiaen (see page 28). A rhythmic canon is one which is based around rhythm alone. It is again based on the principle of strict imitation, but here it is an initial rhythm that is imitated at a specified time interval by one or more parts. Canons are found throughout the opera *Lulu* (1929-1935) by the Austrian composer Alban Berg (1885-1935). In Act I, scene 2, the death of the Painter is accompanied by a five-part rhythmic canon played on unpitched percussion - drums, a tam-tam and a gong.



A more specialised form of the rhythmic canon is the **rhythmic tiling canon.** This is a type where the canon is crafted so that once all the players

have begun to play or sing, there will be exactly one player striking each beat. In other words the rhythms yield one and only one onset per pulse. A basic form of the rhythmic tiling canon can be found in the vocal genre, *katak* music, which is sometimes referred to as the Balinese monkey chant (see pages 46-7).

Fugues

Unlike the canon, where one theme continues from the beginning to the end, the fugue is a more complicated form which follows a detailed set of rules: two or more voices use the systematic imitation of a principal theme (the subject) an answer and (sometimes) a countersubject (a second theme that accompanies the subject), in simultaneously sounding melodic lines (counterpoint).[142] The answer follows the subject in the second voice. It is a statement of the main subject but in a different key, usually the dominant. Statements of the material incorporate contrapuntal devices that alter the subject in some way. These devices are based on symmetry and include those of scale (augmentation and diminution) and horizontal reflection (inversion).

The term 'fugue' has been in continuous use since the 14th century. At first the word held a variety of meanings with imitation being the defining characteristic. However, from the mid-seventeenth century, a theoretical model began to evolve and in 1725, this model was largely set by the Austrian composer and theorist Joseph Fux (1660-1741) in *Gradus ad Parnassum*, a theoretical treatise on composition whose middle sections focus on species counterpoint and fugue. *Gradus ad Parnassum* served as a textbook for several generations of composers and the rules set out were followed by many, Haydn, Mozart and Beethoven amongst them. Central to Fux's treatise are the four rules for voice-leading which are applied to species counterpoint

142 The fugue opens with the subject, the main theme of the fugue. The answer follows the subject in the second voice. It is a statement of the main subject but in a different key, usually the dominant. Some fugues also have a countersubject, a second theme that accompanies the subject. The word episode is used to delineate a passage where no voice states the subject in its entirety. The music in episodes is usually, but not always, developed from the subject or countersubject. Stretto is an overlapping effect where one voice states the subject and is closely followed by another voice before the first one has finished.

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in two, three and four voices before sections on imitation and fugue. The method of instruction is through a dialogue between teacher and pupil and is based on exercises using a given *cantus firmus* for which a counterpoint is to be constructed. Several rules are set up in the first exercise which include:

- all intervals must be consonant
- parallel motion of perfect intervals is forbidden
- perfect intervals must not be approached by similar motion
- disjunct motion by skip must be used sparingly.

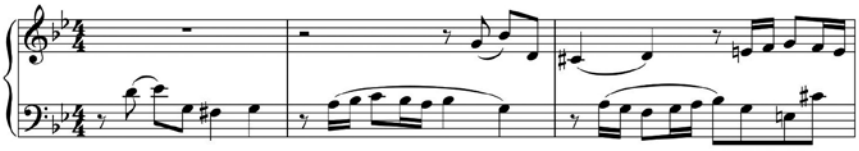
This strict set of procedures has much in common with a mathematical algorithm where a set of precisely described instructions or routine procedures are designed to be applied systematically through to a conclusion in a number of steps.

Gradus ad Parnassum was held in high esteem by J S Bach whose keyboard work *The Well-Tempered Clavier*, a set of preludes and fugues in all 24 major and minor keys was completed in 1742. This collection of fugues is commonly viewed as the zenith of the form (see page 77). By the 1750s the fugue began to be seen as outdated and it no longer enjoyed a central role in contemporary compositions. However, the study of fugue and counterpoint was still an important part of a composer's training and fugal imitation was often found in compositions. Haydn wrote fugues as finales to several of his works, as did Mozart, and Beethoven who adopted a free approach to fugues, notably in the finale to the *Hammerklavier Sona*ta Op. 106 (see pages 68-9) and the string quartet the *Grosse Fuge Op. 133*. Tonality was embedded in the strict rules of the structure of fugues therefore twentieth-century developments in atonality did not lend themselves to the form. Between the 1920s and 1940s however, Neo-classicism revived some of the elements of eighteenth-century musical precepts translated into a modern idiom.[143] Later on both Shostakovitch and Hindemith composed collections of fugues inspired by J S Bach's *The Well-Tempered Clavier*: Shostakovich's *24 Preludes and Fugues* (1950–51) and *Ludus Tonalis* (1943) by Paul Hindemith (1895-1963). *Ludus Tonalis*, subtitled 'Studies in Counterpoint, Tonal Organization and Piano

143 Examples of fugues written in this style include the second movement of Stravinsky's *Symphony of Psalms* (1930) and Bartók's *Music for Strings, Percussion and Celesta* (1936).

Playing', includes twelve three-part fugues.[144]

It should be noted that the formulaic/symmetrical rules of the fugue are sometimes modified to fit the irregular/asymmetrical nature of the major/minor scale. So, for example, in diatonic music, although strictly speaking the answer in a fugue is a statement of the main subject in the dominant (a real answer), the intervals are sometimes adjusted based on the key signature. The melodic contour remains the same, but the transposition is not exact (a tonal answer). So for example in J S Bach's Fugue No. 16 in G minor (from Book I of the *Well-Tempered Clavier*), the subject opens with a rising semitone from D to Eb. If the answer were an exact transposition of the subject, then it would open with A to Bb, however, it opens with a rising minor 3rd – G to Bb, a tonal answer.



As time went on artistic licence increasingly overrode the formulaic approach to the point where in the fugues of Hindemith and Shostakovitch, twentieth-century contemporary musical idioms, (harmonic and rhythm techniques, for instance) are integrated into fugal procedures in ways that are impossible to reduce to formulas.

Fugues, computers and Artificial Intelligence

Composers of fugues followed a set of steps and procedures and as well as adhering to the rules of counterpoint, they also had to understand how to handle the entity of thematic, contrapuntal, and harmonic elements so that even the most complex counterpoint had a firm harmonic basis. In

144 Hindemith, living in the United States, was inspired to compose *Ludus Tonalis* following a then much derided broadcast of Shostakovitch's *Leningrad Symphony*. Hindemith condemned the trend toward "despicable rubbish" in orchestral music and hoped to "remind those who have not succumbed what music and composition really are"cited in Alex Ross. *The Rest is Noise*. (London: Fourth Estate, 2008): 299.

conceiving of the original subject, they needed to bear in mind the answer and the countersubject to ensure that they were a satisfactory harmonic and contrapuntal combination; few could disagree that the design of a good fugue subject is a significant part of the process of writing a fugue. It could be argued that the multi-faceted reasoning employed is similar to that used in solving a mathematical problem. As Joel E Cohen writes of the kinship between mathematics and music 'The processes of creation are similar; that is, in both the most beautiful combination is selected from an infinitude of possibilities and its logical potentials are developed consistently'. He goes on to write 'The end products of mathematics and music depend for their meaning upon the successive relationships of their elements and upon the order imposed by the creator'.[145] To this could be added that, ideally, the end product should be convincing and musical. Given that the set of procedures which need to be followed in order to compose a fugue could be described as an algorithm, perhaps a computer programme could be designed to fulfil this function. As the computer scientist Andrés Garay Acevedo writes of Fux's *Gradus ad Parnassum*, 'such a well-studied process starts to suit the idea of computer assisted composition.'[146] This fact was recognised by the American electronic music composer Laurie Spiegel (b. 1945). In 1981 she delivered a conference paper entitled 'Manipulation of musical patterns' where she argued that a 'working knowledge of all the processes of transformation which can be aesthetically applied' to musical patterns should be important considerations in computer music system design adding that there also needs to be a 'practised awareness of how such materials and operations … relate to and influence each other's' potentials'. She goes on to list relevant materials and operations which include, transposition, retrograde, inversion and scaling.[147] In 1986 Spiegel went on to design her algorithmic composition software *Music Mouse* which she labelled as an Intelligent Instrument with reference to the programme's built-in knowledge of chord and scale

145 Joel E Cohen. 'Some Relationships between Music and Mathematics'. *Music Educators Journal*, Vol. 48, No. 1, 1961: 108-109.

146 Andrés Garay Acevedo. 'Fugue Composition with Counterpoint Melody Generation Using Genetic Algorithms'. *Computer Music Modeling and Retrieval*, Volume 3310, 2005.

147 Laurie Spiegel. 'Manipulation of musical patterns'. In *Proceedings of the Symposium o Small comuters and the Arts*: 19-22. Philadelphia, PA: IEEE, 1981.

conventions, and stylistic constraints. She composed several works using *Music Mouse* including *Three Sonic Spaces* (1991).

One of the first to attempt to create computer-generated fugues was the American musicologist and composer, David Cope, who published *The Well Programmed Clavier* a set of keyboard pieces which he had generated through a computer programme 'Experiments in Musical Intelligence' devised to produce new works in the styles of various composers. The works are discussed in his books *Experiments in Musical Intelligence* (1996) and *The Algorithmic Composer* (2000).[148]

In 2001, an experiment was conducted at Cornell University in New York. The aim of the artificial intelligence project 'Fugue Generation with Genetic Algorithms' was to establish whether the project team could create fugues 'from scratch'.[149] The subject was generated first by the programme and then the answer and countersubjects were generated, following 'certain rules and guidelines'. The first part was 'accomplished through the use of a genetic algorithm' using 'random subjects which were slowly improved 'into something decent'. A 'fitness function' was used to evaluate the suitability of the different features of the subject. This looked at, for example, similarity of pitch where each note was discouraged from being 'different from its predecessor', similarity of note length where notes of similar length were encouraged to avoid 'switching around arbitrarily', and key, where notes were awarded for 'being in the key of C' which is followed by the rather naïve explanation that 'This goes a long way toward making the subject sound better. It removes the majority of sharps and flats from the subject, making it much nicer to listen to.' This set of criteria betrays a lack of musical knowledge in its assumptions. Testing the final product on a team of listeners, the research team came to the conclusion that although the system was 'able to successfully generate subjects, and from there create the beginning of a fugue' it was unable to generate complete fugues, due to the complexity involved. This is not surprising given the naïve assumptions of the fitness function. They found that 'while many of the pieces generated by our system have been rated highly by our test listeners, they are still not competition for a human composer'.

148 David Cope. *Experiments in Musical Intelligence (1996)* and *The Algorithmic Composer (2000)* published by A R Editions, MIT Press.

149 Eric Milkie and Joel Chestnutt, S 473 Artificial Intelligence Project, Fugue Generation with Genetic Algorithms, Cornell University.

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In his paper 'Fugue Composition with Counterpoint Melody Generation Using Genetic Algorithms' (2005), Andrés Garay Acevedo presented the results of the implementation and evaluation of a genetic algorithm which was used to help in the task of automatic counterpoint generation. A fugue subject was used as an input for the system and then a genetic algorithm was used in order to find a suitable counterpoint melody that could be arranged in a basic fugue structure with an answer and two counter-subject melodies arranged over time for three voices. As such, the process was more sophisticated than that used by Cornell University; Acevedo recognised that 'if an algorithm is to be used, then it must incorporate higher-level concepts of musical arrangement and coherence' in order to have 'some sort of aesthetic value.' The algorithm was tested with two different input melodies, and evaluated for fitness: the results were deemed to be satisfactory.

In 2014, a further piece of research was undertaken by Yu Yue Yue Yang and Andrew Horner which sought to improve on David Cope's results and those of Cornell University. 'Automated Fugue Generation' worked on the premise that 'A fugue is a musical composition of multiple voices built on a subject (recurring theme) that imitates itself frequently throughout the piece. The objective was to design and build a system that generated three-voice fugues, emulating the style of J. S. Bach. The subject was to be two bars long and then the position of the new entries was determined bearing in mind the voice, bar, key and 'tonal centre progression'. The characteristics of Bach's fugues were taken into account and the results were evaluated in terms of the frequency of repetition, the dominance of stepwise motion, the range, the link between segments and the fitness of the chord and the pitch structure. The evaluation found that their results 'made musical sense in general, occasionally with unnatural transition and were significantly better than the Cornell results but 'pale' compared to David Cope's pieces. They concluded that they had built a system that could generate polyphonic music similar to fugues, but that it had limitations and the results sounded more like pop music than authentic Baroque music.

It would appear that although the composition of fugues is often referred to as one of the most mathematical and formulaic procedures, it cannot be reduced to these two features; understanding the mathematical structure does not tell us anything about the effect on the audience. Although the algorithm is a partner in the creative process it could be argued that it is no more than that.

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